# THE MATHEMATICS TEACHER

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# THE MATHEMATICS TEACHER

*Devoted to the interests of mathematics in Elementary and Secondary Schools*



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# THE MATHEMATICS TEACHER

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## « Contents »

A Psychological Explanation of Failures of High-School Mathematics ........ *Charles H. Judd* 185

The Construction and Validation of a Test of Geometric Aptitude ........ *J. Murray Lee* and *Dorris May Lee* 193

A Maintenance Program for Tenth Grade Mathematics .. *C. C. Pruitt* 204

In What Respects Should a Child at the End of a Year of Geometry Be Different from What He Was at the Beginning of the Year as a Result of the Training He Receives? ....... *Margaret C. Byrne* 209

Irrational Quantities ........ *J. F. Phillipi* 212

Applications of Complex Numbers to Geometry ...... *Allen A. Shaw* 215

Grading Entirely by Tests ........ *P. Stroup* 227

Blaise Pascal ........ 229

John Wesley Young ........ 232

News Notes ........ 235

New Books ........ 238

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THE MATHEMATICS TEACHER

Volume XXV Number 4

Edited by William David Reeve

# A Psychological Explanation of Failures in High-School Mathematics*

*By* Charles H. Judd, *School of Education*
*University of Chicago, Chicago, Illinois*

This paper is a plea by a psychologist for the treatment of students who are having difficulty in mathematics by the same kind of skillful, scientific methods as are employed in dealing with human beings in need of physical relief. I begin by accusing you of not always being efficient in your work. I have it in mind to present some statistics that I feel sure ought to be of interest to you. I find these statistics one of the best introductions to my talk because I find they always secure attention for what I have to say, when otherwise I should not be able to command attention.

I do not know whether you know the fact that, in competition with the other courses offered in high school, mathematics is suffering and it is suffering greatly. You ought to pay some attention to the fact that there is a distinct falling off in the percentage of the students who are taking algebra and geometry. If you consult the report of the Commissioner of Education for 1890, the first year in which the Commissioner was able to give any definite information on registra-

* Read at the Mathematics Section of the University of Illinois Conference at Urbana in November, 1931.

tions in high-school courses, you find that 45.4 per cent of the students in American high schools were pursuing algebra and 21.3 per cent were pursuing geometry. The contrast between algebra and geometry is striking. The mortality in the first year of the American high school has always been very large. The explanation is to be found in no small measure in the strenuousness of your own department. In fact, I have heard teachers of mathematics say that algebra weeds out the incompetent. If you take that attitude, I think you should be perfectly clear in your mind about the effects of your efforts on pupils. You have, in reality, made algebra a prerequisite for senior literature, or history of the United States, or the social sciences. If you are going to put this hurdle at the beginning of the high-school pupil's career, and if you are going to eliminate him through your algebra, your responsibility is not merely to your own field. You ought to recognize the fact that this responsibility is also to education in general. The very heavy percentage of withdrawals in secondary schools can be traced in very large measure directly to the difficulty in first-year mathematics.

When you look at the statistics for 1900, you find that the percentages have increased both for algebra and for geometry. Algebra has a percentage of 56.3, and geometry has a percentage of 27.4. During the decade from 1890 to 1900, the secondary-school population, in general, was beginning to increase while the curriculum was practically stationary. There was very little new material of the sort that is now enriching the curriculum. In other words, the enlargement during the decade from 1890 to 1900 was chiefly in population. So, also, in the next decade; in 1910 mathematics is holding its own. Indeed, there is a slight increase in the percentage of pupils taking geometry. Algebra has a percentage of 56.9; geometry, of 30.9. Mathematics is a recognized field, and during the first decade of this century there is no serious question about mathematical courses in the freshman and sophomore years. When we examine the figures for 1922, we find the percentage of registrations in algebra which used to be 56.9 is now 40.2 and the percentage in geometry has gone from 30.9 to 22.7. These are percentage figures. One can find some encouragement in the fact that the absolute number of young people taking mathematics in our high school has increased. From 1900 to 1930 there has been a great increase in the secondary-school registrations. In 1900 the number of secondary-school registrations was 500,000. At the present time it is about 4,000,000. The great increase in the number of pupils

results in an absolute increase in the number of those taking algebra and geometry. The interesting part of the situation for our purpose is that in 1922 the percentages have fallen to 40.2 for algebra and 22.7 for geometry. These figures ought to appeal to mathematics instructors. It is very clear that some change is going forward in your field.

I said a moment ago that the foregoing statement with regard to increases in pupil registrations is not merely a statement of a situation in mathematics; it relates to the whole secondary-school curriculum. Why is this drop in percentages of registrations in mathematics courses happening? It is happening, of course, in part, as a result of the competition set up by new courses in the sciences and by new vocational courses. On the other hand, if you go to any high school or to any one of the educational institutions, including the elementary school, you find that the mortality in mathematics is so large that it must be obvious to those dealing with the pupils attempting to take mathematical courses that something is seriously wrong.

I once asked a number of mathematics teachers, with whom I came in contact in a state survey in which I participated, to make a report to me with regard to the reasons why students fail in mathematics. The answers were very vague and general. Some said the pupils were inferior intellectually. Some said that the pupils were unprepared. If you call a physician to your home and he says vaguely, "You're sick," you say, "We'll get somebody else." You do not want to talk in general terms. If you are suffering from typhoid or pneumonia, you are not particularly interested in vague, general terms. You want to know what you have, because treatment will be different for different ailments—so different that any vague diagnosis will be of very little avail. When we find algebra courses that fail 25 per cent of their pupils and, still worse, algebra courses where the failure is 50 per cent, we are facing a social situation which should be faced squarely. Either 50 per cent of the world is not competent to follow in your footsteps or you're not competent to invite them, and you ought to find out which is the true description of the situation.

I can give you some evidence that the pupils who fail in algebra courses in secondary schools and the pupils who fail in arithmetic courses in elementary schools are not entirely to blame. They learn to read, and mentally defective individuals cannot read. They show skill in the practical arts, and this is possible only when people have some brains. Any competent teacher of a high-school class has to face the

fact that he is dealing with normal human beings. If such normal persons are not able to carry forward tasks assigned, it is someone's responsibility to see that there is a sufficient adjustment between tasks and pupils so that there shall not be an abnormally large percentage of failure on the part of normal individuals.

We have made careful surveys in many instances of young people who are failing in the first course in high-school mathematics or in some of the courses in arithmetic, and the interesting fact is that they are not failing in all phases of the courses in mathematics. They fail at certain points in the course. If you could locate the points at which they are failing, you would have the key to a very large share of the difficulties, because when a student in mathematics has failed at a certain point, he is very likely from that period on to be in difficulty with the various ideas which appear later in the course. If you fail utterly to understand a certain fact on a certain page in one of the social sciences, you can go forward and begin over again at another point. You can carry on a type of reasoning which will restore your ability to deal with the subject. There is, on the other hand, in mathematics so close an interdependence of topics that recovery at a later stage in a course is impossible. You can find a pupil carrying on up to a certain point in arithmetic, or in algebra, or in geometry, and then he gets off the track. Once he gets off the track, he is in difficulty of a type which has serious consequences for all of his subsequent efforts in that field. He will fail unless somebody aids him to get back on the right track.

The problem which I am trying to present is a psychological problem. My plea to you is that you consider your students as psychological beings, not merely as mathematical failures. May I offer an illustration borrowed from geometry? We have a figure which is commonly employed in the textbooks on geometry when we begin to teach pupils the relation between two similar triangles. We use very simple figures and we place the two triangles in positions so that their similarity is a striking fact; it stands out as clearly as we can make it stand out. We might draw the same triangles in positions in which their similarity would not be obvious. If we present the figures to students in positions which make recognition of similarity easy, we do not give them very useful training for later understanding of complex situations. If you want your student to be keen in complex situations, to see the relation between constructive and demonstrative geometry, you should not simplify your demonstration so much. You

simplify mathematics but you do not educate the student of mathematics. You do not give him the kind of contact with the mathematical principles that will give him intellectual independence.

I hear high-school teachers say that students are morons. The fact of the case is that the student who has sufficient intelligence to deal with the other natural sciences, who has sufficient intellect to deal with literature and art, is not a moron. You have undertaken to turn ordinary normal intelligence into a mathematical intelligence, and the responsibility for success is not alone with your students. I am disposed to place the blame on you if you do not succeed in doing the particular thing you undertake to do.

As a further example of psychological analysis, let me take a very simple illustration. If I try to add a long column of figures, I get along well for a time, but there is a region in the neighborhood of the 70's where I have to pause before I take the next jump. I have to stop and say "seventy-six" several times and hold on for a few minutes before I attempt the next addition. I am trying to get my second intellectual wind. I know perfectly well the relation between 76 and 8, but I am literally out of breath and I have to get some air. Put in psychological terms, I have reached the limit of my span of attention. The span of attention can be lengthened only slowly and as a result of much practice.

Let us discuss the span of attention and algebra. Students learn from you that if they multiply two minus quantities they will get a certain sign in the end. They can recite the rule at any time; they can tell you glibly what sign should be given to the result of multiplying two positive or two negative quantities. If you give them the rule and an example, they can solve the problem. If, however, you complicate the situation by introducing complex factors, parentheses, and other difficult elements, you go beyond the pupil's span of attention. He does not recall the rule and he misses the sign at the end because his attention has been overtaxed. Recall what was said a moment ago: The span of attention can be lengthened only through long practice. You have in algebra a whole series of items that have to be thought of and you rush through many rules in a week or two. If your pupils can recite the rules, you are too apt to think they know algebra. The teaching of mathematics moves too rapidly. You misjudge the rate because you have been over it before. When your pupils are introduced to a novel situation made of five or six elements, they fail. What you have to say to yourself is that a recon-

struction of the mathematics course has to be undertaken in terms of psychology.

The plea I am making with you is to recognize these perfectly obvious facts. No 25 per cent or 50 per cent of a population is deficient. There is no fact that gives you any right to call that many high-school pupils deficient. You cannot berate a whole generation. If you do, they will leave you. The fact of the case is that any intelligent population is wont to go away from something that is not properly handled. You should note in this connection that the percentage of pupils taking mathematics is falling off very rapidly. The statement I am here to make is that you cannot conduct mathematics in such a way that it does not satisfy the majority of people. If you do, you are going to withdraw from our civilization one of the most important forces for mental development and training. I am so much in sympathy with mathematics that I believe my plea is justified when I ask you to improve the teaching of these sciences which are of the greatest significance to our generation.

There is a branch of psychology which studies the emotions. Perhaps you will say that emotion is entirely outside of the domain of mathematics. It becomes my duty as a psychologist to point out that every student who comes into your class comes with an emotional bias which is so significant for his intellectual operations that you have to give some heed to it. This emotional bias has often been created by traditions resulting from the contact of earlier groups of students with your subject. Students are afraid of you. Were you ever afraid of anything? I might go into detail and tell you that the emotion of fear was of great importance to the race when it was living in primitive surroundings. Fear was planted in human nature as a device for protection. It is a very definite and vivid fact. A number of glands participate when you are afraid. The only difficulty is that when the glands begin to operate in many modern situations their action is disadvantageous. Human nature has evolved itself out of the primitive stage, and fear, from the social point of view, is one of the worst things one can have because it interferes with one's actions. Fear is a thing which is just as definite and real as the circulation of the blood and is just as important in determining what one is going to do next. Your pupils are afraid. They have been told by all the former students that about 25 per cent of them are certain to fail. They are afraid of algebra, and when mathematics sud-

denly becomes geometry, they are afraid again. Do you explain why algebra and geometry belong to the same category? Geometry does not look like the algebra course they have just finished. They wonder whether they have to go through the whole ordeal again with pictures instead of letters. The teacher whose class is afraid ought to understand that he is dealing with human nature and that frightened human nature has to be reassurred.

I went over to our high school some years ago and said, "I should like to find somebody who is failing in mathematics so that I can make a psychological study of the case." I sat down with the boy who was supplied to me. He was having the experience that 25 per cent of the world has. I asked him where the lesson was, and his answer was, "I don't know." He had reached the point where it didn't make any difference where the lesson was. After some research we found out that the lesson was on page 137, and we opened the book to that page, and the boy looked at it with a blank expression on his face. I said, "Did you ever see anything like that in the book before?" "No," he replied. I said, "Isn't it reasonable to suppose that if these problems are given on page 137 there must be something in the earlier pages that will help us in their solution? Let us look over the first 136 pages and see if we can find help." Did you ever have a pupil give you a look which seemed to say: "Read 136 pages in less than half an hour? I'm willing to fail in algebra if I have to deal with 136 pages in a few minutes."? Such pupils do not know how to go back and find the method of dealing with a difficult point. Every pupil who uses a textbook ought to know that the textbook contains a succession of exercises leading forward. When you come to the end of one chapter and jump into another which seems to be altogether different, why not stop and explain mathematics to the pupils?

I do not know of any course where there is more misery because of lack of understanding than there is in mathematics. I speak as a layman. When I studied algebra, I used to see some of the members of the class perform operations in factoring that were marvelous. Of course, teachers were able to factor with even more astonishing agility. Nobody introduced me to factoring in any proper way. It was one of those performances which you did or didn't do. Whatever you did was done by chance. If you could juggle the figures backward and forward, you might arrive or you might not. I consumed much midnight oil trying to get the combination which I had to have fac-

tored ready for the next day. My chance of getting it was not more than 25 to 50.

If you are going to put children through factoring, and I suppose you have to, I say as a friend of children that you ought to teach the *why* and not merely the *what*. I have seen a great many high schools in my day and I have visited a great many classes in algebra and geometry. I believe that there is too little explanation and too much demand that pupils supply the explanation at the difficult points. My plea to you is that you make enough tests so that you know where the difficulties are. Help the student to see how to meet difficulties; help him when you see that he is losing out. Mathematics should be an explanatory subject and not merely an exacting demand on your pupils.

# The Construction and Validation of a Test of Geometric Aptitude

*By* J. Murray Lee
*Director of Research, Burbank, California*
*and*
Dorris May Lee
*Glendale City Schools, Glendale, California*

Is there a need for an instrument of guidance in the field of geometry? The answer from practically all school people who have had to meet the problem is, "Yes." Geometry was placed in our course of study when the almost sole function of the school was to prepare for college work. It was expected that all pupils should take it as it was considered necessary to their future studies. Furthermore, the students at that time were a highly selected group, only those of high intellectual ability attempting this advanced work. Thus it was not unreasonable to expect all to study geometry with at least a moderate degree of success.

How different the situation is today! Only a small percentage go to our colleges, while our high schools include nearly all pupils of high school age. We cannot expect all these to succeed in as academic a subject as geometry, nor should we feel that it is necessary for all to study it, even if they were capable of succeeding. This means, then, that guidance is necessary. Not only is it required but it is being practiced in every high school in the country today. Usually it takes the form of self-guidance on the part of the pupil from information he has gathered, by remarks of other students or by what other students are taking, by his grade in algebra, which subject he must have passed before he may take geometry, and by college requirements. Are these sufficient or are they sound?

If we could have a measure which would pick out the majority of the pupils who would fail in geometry, or at best receive just a passing mark, and then by conference with the student talk over his chances of passing or getting a recommended grade, could we not steer many pupils into something which would be much more worth while for them both in information received and in self-respect gained by success rather than failure?

## Description

A description of the construction and use of such a test is the purpose of this article. In selecting the material, the results of previous studies dealing with guidance in geometry led to the hypothesis that by using psychological tests dealing with numerical relationships and spatial concepts a much better basis for guidance could be obtained. Following this plan a battery of eight tests were constructed. Samples indicating types of material used, are given below.

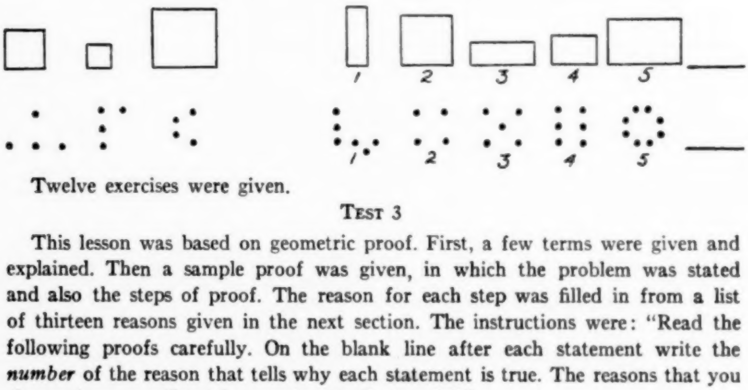
### Test 1

Solve the following equations for $x$:

1. $ax = ac + ab$
6. $x + 10 = 3(x - 2)$
12. $\dfrac{a}{x} = \dfrac{b}{c}$

### Test 2

In this test the pupil was to find the figure from the last five which was most like the first three and place the number under it in the margin.




Twelve exercises were given.

### Test 3

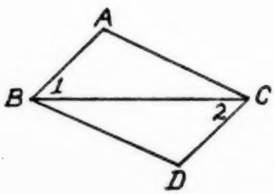
This lesson was based on geometric proof. First, a few terms were given and explained. Then a sample proof was given, in which the problem was stated and also the steps of proof. The reason for each step was filled in from a list of thirteen reasons given in the next section. The instructions were: "Read the following proofs carefully. On the blank line after each statement write the *number* of the reason that tells why each statement is true. The reasons that you choose from are listed by number in Section III."

Proof 1. Given: $AB = CD$

$\angle 1 = \angle 2$

To prove: $\Delta\ ABC$ is congruent to $\Delta\ BCD$

| Statements | Reasons | |
|---|---|---|
| $AB = CD$ | Why?...................... | 1 |
| $\angle 1 = \angle 2$ | Why? ..................... | 2 |
| $BC = BC$ | Why?...................... | 3 |



Therefore $\Delta ABC$ is congruent to $\Delta BCD$. Why? .................... 4
Five proofs were given which had a total of 24 pupil responses.

### Test 4

Three means of finding the value of angles were given, namely, (1) the sum of all the angles of a triangle equals 180°, (2) when two straight lines intersect, the vertical angles are equal, and (3) the sum of the successive angles around a point on one side of a straight line is 180°. Twelve problems of finding the values of specified angles in three figures were given.

2. In Fig. *B*, the sum of $\angle 5$, $\angle 6$ and $\angle 7 =$ ———

6. In Fig. *B*, $\angle 5 + \angle 6 = 145°$, $\angle 7 =$ ———

9. In Fig. *B*, $\angle 5 = 60°$, $\angle 6 + \angle 7 =$ ———

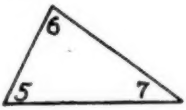


Fig. B

### Test 5

The prices of various fruits and vegetables were listed. The directions were, "Supply the missing numbers in the following problems as shown in the samples":

1. 7 lbs. of grapes cost —— as much as 4 lbs of apples.
7. 1 lb. of peaches cost —— as much as 1 lb. of cherries.
14. 2 lbs. of cherries and 2 lbs. of prunes cost —— as much as 3 lbs. of peaches.

Fourteen problems were given and any correct answer was credited.

### Test 6

The designation of geometric figures by letters was explained. The definition of perimeter and the fact that in a parallelogram the opposite sides are equal as well as parallel were given. There were twelve questions as to the length of lines or perimeters in a given figure, the values of some of the lines being supplied, the problem is to deduce those required.

1. How long is *HF?*
8. What is the perimeter of the parallelogram *FBCD?*

### Test 7

This test was a series of twelve problems involving literal expression of figures in two and three dimensions.

1. What is the area of a board $l$ feet long and $w$ feet wide? ——
7. From a piece of cloth $C$ feet long and $D$ feet wide a piece $A$ feet long and $D$ feet wide is cut. What is the area of the remaining piece? ——
11. If two cubes, the edge of each $e$ feet long, are placed side by side, what would be the volume of the resultant rectangular solid? ——

### Test 8

Here were given drawings of nine piles of blocks, varying from 8 to 61 in a pile. The number in each pile was to be determined by inspection, considering those unseen as well as those in view.

## Validation

The complete battery of eight tests was given to about 600 students in and around Los Angeles in September 1929.[1] These students were just beginning the study of geometry. In January 1930, the same students were given a comprehensive geometry achievement test. Each test in the battery was correlated with the Renfrow Diagnostic Test in Plane geometry, Test I, and with each other.[2] Table I gives the correlations between each test and every other test and with the criterion.[2] By using the multiple regression technique, a group of four tests was selected as giving the best prediction of a pupil's success in geometry. The selected tests were those numbered 4, 5, 6 and 7. Test I had been previously discarded because there was too great a difference in the scores of those who had just finished algebra and those for whom a summer had intervened. Thus, if the test had been left in, the battery would had to have been given at the end of the algebra course rather than at the beginning of geometry. Correlations were also found with school marks in geometry at the end of the first semester, and with the pupil's age at the beginning of the course.

Table I

*The Correlation with the Criterion and with Each Other, of Each Factor Used to Predict Success in Geometry.**

| Test | Criterion | Test 2 | Test 3 | Test 4 | Test 5 | Test 6 | Test 7 | Test 8 | Marks |
|---|---|---|---|---|---|---|---|---|---|
| 2 | .299 | | | | | | | | |
| 3 | .310 | .240 | | | | | | | |
| 4 | .438 | .266 | .530 | | | | | | |
| 5 | .528 | .219 | .219 | .332 | | | | | |
| 6 | .479 | .211 | .295 | .416 | .431 | | | | |
| 7 | .549 | .153 | .200 | .371 | .478 | .522 | | | |
| 8 | .344 | .184 | .147 | .300 | .219 | .271 | .361 | | |
| Marks | .796 | .279 | .325 | .407 | .450 | .447 | .555 | .306 | |
| Age | −.197 | −.173 | −.047 | −.019 | −.213 | .046 | −.131 | .013 | −.239 |

* Based on 200 cases.

The most extensive and hence the most reliable study was carried out in Glendale Union High School, Glendale, California, where complete records on 135 cases were available. Here the correlation be-

[1] The testing included 15 teachers in 5 High Schools located in 3 different school systems. This provided a sampling of varied objectives, courses of study, textbooks and methods of teaching.

[2] *The Renfrow Diagnostic Test in Plane Geometry, Test I.* C. A. Gregory Company, Cincinnati, Ohio.

tween the weighted score of the selected tests and the geometry achievement test was $.720 \pm .028$. It is interesting to note that the correlation between the final achievement test and the semester marks was only .789. Thus, we might say that the Test of Geometric Aptitude foretells a student's achievement almost as well as his achievement foretells his mark.

It is essential to the interpretation of results that the tests be kept exactly as they were when given to the group on which they were standardized. Changing the tests by adding additional items, changing the time limits, or adding additional tests to the battery changes the results so that it is impossible to tell how effective any of the material would be. The four tests for the final battery have been kept in the exact form in which they were used in determining standards. This makes possible direct use of these results in interpreting the student's ability.

The validity of the test is determined by how well the Test of Geometric Aptitude correlates with success in geometry. Table II presents these figures for all the schools included in the standardization.

TABLE II[3]

*Data on Validity*

| | Schools | | | | |
|---|---|---|---|---|---|
| | A | B | C | D | E |
| Number of cases | 135 | 71 | 53 | 106 | 84 |
| True validity | .745 | .735 | .662 | .658 | .633 |
| Correlation between Test of Geometric Aptitude and achievement test+geometry mark | .700 | .649 | .627 | .623 | .600 |
| Correlation between Test of Geometric Aptitude and achievement test | .720 | .701 | .675 | .613 | .645 |
| Correlation between Test of Geometric Aptitude and geometry marks | .637 | .530 | .477 | .607 | .514 |
| Correlation between achievement test and geometry marks | .789 | .640 | .814 | .813 | .818 |

The true validity was found from the expression

$$\frac{r\,(\mathrm{a+m}) \cdot \mathrm{T\,G\,A}}{\sqrt{\dfrac{2r_{a\,m}}{1+r_{a\,m}}}}$$

[3] Dorris May Lee and J. Murray Lee, *Manual of Directions for Lee Test of Geometric Aptitude.* Southern California School Book Depository, Los Angeles, Calif., 1931.

where "a" is the achievement test, "m" is the geometry mark, and "TGA" is the Test of Geometric Aptitude.[4] This is the theoretical coefficient of correlation between the Test of Geometric Aptitude and a true measure of geometric success. However, the nearest this situation could be reached was to average the achievement score and the mark. The resulting correlations are from .600 to .700.

The reliability coefficient of the test was computed by correlating the "odds" and the "evens"; then the Spearman-Brown prophecy formula was used. This gave a correlation of $.911 \pm .011$. This is the theoretical coefficient of correlation that would be obtained by correlating the scores made on this form of the test with the scores made on a similar form of the same test. Table III gives additional information[5] on the reliability of the test.

TABLE III
*Data on Reliability*

| r | N | S.D. | $P.E._{score}$ | $P.E._{\infty \cdot 1}$ | $\frac{P.E._{score}}{S.D.}$ | $\frac{P.E._{\infty \cdot 1}}{S.D.}$ | Nature of group |
|---|---|---|---|---|---|---|---|
| .911 | 107 | 13.24 | 2.66 | 2.54 | .201 | .192 | Unselected 10th grade |

The last figure gives the ratio of the probable error of an estimated true score, ($P.E._{\infty \cdot 1}$) to the standard deviation of the distribution of obtained scores (S.D.) The interpretation[6] of this ratio is that ratios less than .20 indicate a sufficiently reliable test score for individual diagnosis.

## USES

There are four main uses of the Test of Geometric Aptitude. Perhaps the most important of these is to determine whether or not a student should take geometry. To provide a basis for this a critical low score of 24 has been set, and students getting below this should be counseled against taking geometry. This score was set, on the results of the performance of 450 unselected cases in five schools. The method of determining the location of this score was to determine the

[4] C. L. Hull, *Aptitude Testing,* World Book Company, 1928, p. 244.

[5] G. M. Ruch and George D. Stoddard, "Tests and Measurements in High School Instruction," pp. 355-374. Yonkers-on-Hudson, New York: World Book Co., 1927.

[6] Ruch and Stoddard, *op. cit.*, p. 27.

percentage of students at each level receiving marks below C. From an examination of the data no consistency was found as to the percentage of failures at different levels as is found in the case of algebra, but when the "D" group was added the percentages showed the expected gradual increase at successively lower levels. As a check on the above procedure, it was found that the lower limit of the achievement scores was the same for both the "D's" and the failures, and the achievement scores of 73 per cent of the "D's" were included in the range of scores of the failures. The conclusion was reached that the decision between giving a pupil a "D" and a failure depended on other factors than success in the subject. For instance, a pupil would be much more likely to attempt to make up a failure than he would a "D" and teachers, feeling that a repetition of the course would be futile give the "D." It is also true that it is more difficult to get as objective a basis for grading in geometry as it is in algebra.

Table IV shows the number of students failing and the number of "Ds + failures" at each level and the percentage they are of all those receiving scores in that level. This shows very clearly the location of the critical low score, since below this score more than 2/3 of the

TABLE IV[7]

*Number and Percentage of Failures and "D's+Failures" at Each Interval*

| Score | Number of Failures | Per cent of Failures at Each Level | Number of D's+ Failures | Per cent of D's+ Failures |
|---|---|---|---|---|
| 68 + | | | | |
| 64–67 | | | 2 | 11.1 |
| 60–63 | 1 | 3.0 | 4 | 12.1 |
| 56–59 | | | 3 | 8.3 |
| 52–55 | 2 | 4.9 | 4 | 9.8 |
| 48–51 | 2 | 6.5 | 4 | 12.9 |
| 44–47 | 2 | 4.9 | 6 | 14.6 |
| 40–43 | 3 | 7.3 | 6 | 14.6 |
| 36–39 | 5 | 10.6 | 14 | 29.8 |
| 32–35 | 2 | 7.1 | 11 | 39.3 |
| 28–31 | 8 | 32.0 | 11 | 44.0 |
| 24–27 | 5 | 14.7 | 17 | 50.0 |
| 20–23 | 7 | 36.8 | 13 | 68.4 |
| 16–19 | 9 | 50.0 | 15 | 83.3 |
| 12–15 | 8 | 44.4 | 13 | 72.2 |
| 8–11 | 2 | 66.7 | 3 | 100.0 |
| 4– 7 | | | 1 | 100.0 |

[7] If interested in similar comparisons in algebra see the Manual of Directions for the Lee Test of Algebraic Ability, Public School Publishing Co., Bloomington, Ill., 1930, Tables 5-8.

pupils receive "Ds" or failures and more than 1/3 receive failing grades. Above this score more than 1/2 receive a "C" or better, and at only one level do more than 15 per cent fail.

The next step was to study the records of the students getting below 24 that did *not* fail. A total of 59 students received below 24. Twenty-six of these students failed and 33 passed. Out of the 33 that passed 19 made "Ds" and the teachers would have advised 9 out of the remaining group not to take geometry. The remaining 5 made the following marks:

TABLE V

*Marks of Students Passing with Scores Below 24 Who should have taken Geometry According to the Judgments of their Teachers*

| Grade | Number | Per cent of Group Below 24 |
|---|---|---|
| B | 2 | 3.4% |
| C | 3 | 5.1% |
| Total | 5 | 8.5% |

This indicates that only 8.5 per cent of the group that would have been eliminated through the use of the "critical score" should, according to their teachers, have taken geometry. These judgments were made by the teachers after having taught the pupils for one semester. The two pupils who received "Bs" and two of the three who received "Cs" were all from one school so this would seem to indicate that there is really a smaller chance for a pupil to succeed whose score is below 24 than the figures would indicate. Also, there is no evidence that these students would not have received as much or more benefit from spending the same amount of time on some other subject for which they were better adapted.

The next question is, how many failures will be left if the proposed scheme is followed? Using the data on the same group, there were 56 failures in the first semester of geometry, 26 of whom would

TABLE VI

*An Analysis of Students Receiving Scores Below 24*

| | Number | Per cent of group below 24 | Per cent of total group |
|---|---|---|---|
| Failures | 26 | 44.1 | 5.8 |
| Passing | | | |
| Should not take | 28 | 47.5 | 6.2 |
| Should take | 5 | 8.5 | 1.1 |
| Total | 59 | 100.1 | 13.1 |

have been eliminated by the use of the Test of Geometric Aptitude. The causes of failure of the other 30 as analyzed by the teachers, are set forth in Table VII.

TABLE VII

*Causes of Failure of Pupils with Scores Above 24*

| Failure due to | Number | Per cent of the failures | Per cent of total group |
|---|---|---|---|
| Application | 13 | 23.2 | 2.9 |
| Ability | 10 | 17.9 | 2.2 |
| Absence | 1 | 1.8 | .2 |
| Miscellaneous | 1 | 1.8 | .2 |
| Not accounted for | 5 | 8.9 | 1.1 |
| Total | 30 | 53.6 | 6.6 |

In all probability, the remaining 6.6 per cent of failures could be reduced, since the teachers would have a reliable measure of the student's ability and would be more certain whether the student was not trying or not capable of doing the work.

Although 24 is recommended as the deciding mark, it should be recognized that all borderline cases, i.e., students scoring around 24, should be very carefully scrutinized and more information obtained if possible. The records of students receiving scores from 24 to 31 should also be studied. Probably the most significant facts beside the test score are the student's grade in ninth grade mathematics and his study habits, whether or not he is industrious. *No absolute* standard can be set up, for conditions vary from school to school, and no judgment should be formed until all available data have been considered.

The second use of the test is to section classes. If it is desired to group the students into homogeneous groups, the score on the test will be very effective. If 100 students are to take geometry, the following classification might be made: the 45 or 40 having the highest scores could be put in the fast-moving section, the next 35 in the average section and the remaining 20 to 25 in the slow-moving group.

In case conditions do not permit sectioning and it is desired to differentiate between the work of the students, the test will prove effective in determining the various levels of ability.

The third value of the test is to determine pupils of outstanding ability. It is very desirable to know at the beginning of the year those students whose ability in geometry is outstanding. Too often our most

capable students do mediocre work. Since they can get passing grades with less effort than the others, they form poor study-habits and attitudes and do work far below what they could do. If, at the outset, either the more capable group or the more capable students in the group were given an introduction to supplementary or creative work, they would be stimulated to do more nearly the type of work of which they are capable. Those who can, should be urged to quickly master the routine work of the course and spend the remainder of their time in working on problems or projects which the knowledge of geometry has opened to them. This will accomplish two results: first, the pupil will derive more good from geometry itself, will broaden his general knowledge, and by reporting his work to the class will stimulate others; second, he will have gone a step toward learning how to make the best use of his time and will have discovered the possibilities of individual work.

The fourth use of the test is to provide a measure of ability. Achievement depends upon ability modified by application. One outstanding problem at present is whether low accomplishment is due to lack of ability or lack of application. By having this measure of ability, it will be possible to further decrease the amount of failures due to application. The correct use of this test will result in an increased knowledge of the teacher concerning the pupil and an early diagnosis of the case. The remedy may then be applied before it is too late.

In using the results of the test, it must be remembered that no test is a perfect measuring instrument. Instead of being a final measure of the pupil, it should be the basis for further study. It will, however, prove most valuable as an aid to selection of students and effective teaching.

*Summary*

The special features of the Test of Geometric Aptitude are:

1. It covers only geometric instead of general mathematical ability.

2. The test is given before the student begins the study of geometry.

3. The time required for giving is at a minimum due (1) to the selection of a few tests which are most effective in predicting success, (2) to the simplified directions which can be mastered by any teacher in a few minutes.

4. It requires very little time and expertness to score. The fact that the answers are mostly numbers, the only correct answer is the

one given in the key, rights only are counted, and all answers are spaced to conform with the pupil responses, all help to cut down the time of scoring and make it possible for clerical help to be used.

5. The use of the actual weights obtained from the multiple regression equation in finding the pupil's final score makes possible a minimum time to be spent in testing and scoring with the optimum results.

6. The validity of the test as determined by a correlation with achievement tests is $.720 \pm .028$.

7. The reliability of the test is $.911 \pm .011$.

8. Using the results properly will not only eliminate most of the failures and the resulting discouragements but, through elimination of failures will save the school system considerable from a financial standpoint.

The specific uses to which the results of this test can be put are as follows:

1. To determine whether or not a student should take geometry. Carefully worked out standards are provided as a basis of selection.

2. To aid in sectioning classes. In schools of sufficient size it is desirable to be able to form homogeneous groups and adapt the work to the various levels by differentiated methods and materials.

3. To determine various levels of ability within the class, as an aid to instruction in case conditions do not permit sectioning.

4. To discover at the beginning of the term, pupils who are capable of doing exceptional work in geometry.

5. To provide a measure of the student's geometric ability so that a teacher can tell whether poor work is due to lack of ability or to other factors that can be corrected.

## National Council Members

We will appreciate it greatly if members of the National Council wil send in news items of general interest to our readers concerning meetings of local mathematics clubs, meetings of mathematical organizations affiliated with the National Council, deaths of members, promotions or changes in teaching position, programs of outstanding meetings and the like. Send such notes to Dr. Vera Sanford, School of Education, Western Reserve University, Cleveland, Ohio or to The Mathematics Teacher, 525 W. 120th Street, New York City.

The Editor

# A Maintenance Program for Tenth Grade Mathematics

*By* C. C. Pruitt, *Central High School, Tulsa, Oklahoma*

Probably no subject in the high school curriculum is receiving more attention today than that of plane geometry in the tenth grade. Much of this attention is directed towards the possibility of fusing plane and solid geometry into one course. From this situation, one would infer that all is not well in either the field of plane geometry or that of solid, with probability in both. I think all teachers of mathematics in the senior high school are agreed that the teaching of plane geometry has not advanced to the point where we are satisfied with the results obtained.

One of the greatest fallacies and weaknesses of the mathematics curriculum has been the neglect and abandonment of the skills learned in the preliminary courses. We find that many of the skills and facts learned in the elementary grades in arithmetic are forgotten, or at the most only occasionally brought up, in the later grades. This same process is carried on throughout the senior high school.

The basic skills in the addition of whole numbers are learned in the second and third grades; fractions are emphasized in the fifth and sixth grades; the simple equation in the ninth grade, and so on with each topic through the entire length of the mathematics courses. In many instances, these skills are only first learned through long and tedious hours on part of both teacher and pupil.

Now the tragedy of the whole story is that the same unit of work which required such a tremendous amount of energy in the first learning process is carefully laid away on the shelf and practically neglected during the remaining grades. What is the result? We find that as the pupil advances in grades, these skills that came easily to him in the lower grades are causing no small amount of concern. If he is taking algebra, we find that he has trouble in addition of whole numbers, while even to mention fractions or percentage is likely to cause nervous prostration.

And we, as geometry teachers find the same things to be true with all of the mathematics learned in previous courses. How many times have we shuddered, and often bitterly attacked the teacher

of algebra, when one of our pupils annihilates a formula of the type $A = ab/2$. Again how much success do we have in attempting to give the pupils a problem involving the fractional, or quadratic equation. For my own part, I have often been amazed at the almost total ignorance of exceptionally good geometry pupils along these lines.

Have our pupils forgotten their algebra and arithmetic? Yes. Are they to blame? No. Then wherein does the fault lie? Is it because of poor and inefficient teaching in the lower grades? No, we all know that some of the finest and most capable teachers are to be found in the lower grades. We must look to another source for the root of much of the trouble. I believe that it can be traced to a lack of proper maintenance of those skills. With a proper maintenance program, extending throughout the entire mathematics curriculum, I believe a part of our trouble in presenting plane geometry and other senior high mathematics courses can be greatly reduced.

This is one of the main features which is underlying the new course of study in the Tulsa (Oklahoma) system. A series of twelve drills were worked out for the tenth grade. The drills are to be given at intervals of three weeks. There are 240 questions in the series—20 per drill. Many have two or more parts. Eighty-seven of the questions are purely arithmetical in character, and 153 in algebra.

All of the skills involved in the questions are of the higher decades, and usually combine several of the lesser ones. It would take too long to list all of the different skills involved in each problem, the technique involved in its construction, etc.* However, I shall give a list of the types of problem found in these drills, along with one complete drill.

Arithmetic

| Whole numbers and decimals | No. Problems |
|---|---|
| 1. Addition | 6 |
| 2. Subtraction | 6 |
| 3. Multiplication | 6 |
| 4. Division | 6 |
| Fractions and mixed numbers | |
| 1. Addition | 4 |
| 2. Subtraction | 6 |

* They are available in mathematics office, Central High School, Tulsa, Oklahoma.

| | |
|---|---|
| 3. Multiplication | 7 |
| 4. Division | 7 |
| Denominate numbers | |
| 1. Addition | 3 |
| 2. Subtraction | 2 |
| 3. Multiplication | 4 |
| 4. Division | 4 |
| Percentage (3 cases) | 21 |
| Statistics | 5 |
| | 87 |

ALGEBRA

| | |
|---|---|
| Directed numbers | |
| 1. Addition | 2 |
| 2. Subtraction | 2 |
| 3. Multiplication | 1 |
| 4. Division | 1 |
| Monomials | |
| 1. Addition | 3 |
| 2. Subtraction | 4 |
| 3. Multiplication | 3 |
| 4. Division | 4 |
| Polynomials | |
| 1. Addition | 3 |
| 2. Subtraction | 6 |
| 3. Multiplication | 5 |
| 4. Division | 5 |
| Special products | |
| $(a+b)(a-b)$ | 2 |
| $(a-b)^2$ | 2 |
| $(a+b)^2$ | 2 |
| $(ax+b)(cx+d)$ | 6 |
| | 12 |
| Factoring | |
| $ac+bc$ | 2 |
| $a^2 \pm 2ab+b^2$ | 2 |
| $a^2-b^2$ | 2 |
| $ax^2+bx+c$ | 6 |
| | 12 |
| Powers | 4 |
| Formula | 18 |
| Evaluation | 6 |
| Symbolism and vocabulary | 11 |
| Fractions | |
| 1. Reduction | 4 |
| 2. Addition | 2 |
| 3. Subtraction | 3 |
| 4. Multiplication | 2 |

| | |
|---|---|
| 5. Division | 3 |
| Simple Equation (Non-Literal) | 6 |
| Simple Equation (Literal) | 10 |
| Quadratic Equation | 7 |
| Fractional Equations | 9 |
| Equations (Two Unknowns) | 5 |
| | 153 |

## Test 7

1. Find the average.

656
973
987
492
685
548

Ans. ______

2. Jack said the product of 6879 and 576 is 3692204, Grace said the product is 3962304. Who was correct? If neither is correct, write the correct answer.

3. $3\frac{3}{4} \times \frac{2}{3} \times 1\frac{1}{3} \times 8/9 =$

4. Perform the operation indicated.

(a) $\frac{2}{3} \div 5 =$ Ans. (b) $\frac{3}{4} \div \frac{1}{4} =$ Ans.

5. How much less is 3 yd. 2 ft. 9 in. than 7 yd.? ____________

6. 100% of $N = 8746$. $7\frac{1}{2}$% of $N =$ ____________

7. Mr. Williams, an automobile salesman, receives a commission on his sales. One month his sales were \$2,845 and his commission came to \$142.25. What rate of commission did he receive?

8. What example below is the correct one to use in solving this problem? What per cent of 156 is 11.7?

| (a) | (b) | (c) | (d) | (e) |
|---|---|---|---|---|
| 156<br>11.7 | $\frac{156}{11.7}$ or $11.7\overline{)156}$ | $\frac{11.7}{156}$ or $156\overline{)11.7}$ | 156<br>.117 | 156<br>117 |

9. Add +5, −8, −9, +10, 6, −3

10. $(-x^2y)+(2x^2y)+(-7x^2y)+(x^2y)+(-3x^2y)=$

11. $-ab\overline{)4a^3b^3-3a^2b+2ab^4}$

12. What must be added to $5h^2+8L-7w-9v$ to give $2h^2-8L+4w-3v$?

13. (a) $(3a+2)\ (4a+5)=$

(b) $(c+7)\ (c+7)=$

14. Factor completely.

(a) $b^2+b+\frac{1}{4}$

(b) $w^2-2w-35$

15. Find the sum. $\frac{2c}{c-1}+\frac{3c-4}{c^2-1}+\frac{5}{c+1}$

16. Solve for $m$. $5cm-dm=10C-2d$

17. In the formula $C = 5/9\ (F-32)$, find the equivalent Centigrade reading, when a Fahrenheit thermometer registers 77°.

18. (a) In the term $3a^2x$, the number of literal factors is __________. The number of numerical factors is __________. (b) To the term $Z$ supply the numerical coefficient 4, the literal coefficient $m$ and the exponent 3. Ans. __________$Z$

19. Solve for $t$ $2t^2 = 5t$

20. Solve for $m$ $\dfrac{m-4}{m-1} = \dfrac{m-2}{m+3}$

Six of these drills were given the first semester of the present school year (1930–1931). After the drills are given the types of problems missed are discussed. If we feel that a pupil needs remedial work on any particular type of unit, special drills are given him. Remedial drills are provided for all units of skill involved in the drills. If the theory of distributed practice holds true these students are kept from forgetting their algebra and arithmetic by these drills.

While the time taken for administering and discussing these drills breaks in on the time for geometry, the results certainly have justified the time spent. Notwithstanding the fact that we have only begun this procedure in our geometry classes decided improvement is already noticeable in the way pupils handle exercises involving algebra and arithmetic. It is no longer necessary to review multiplication of fractions, simple equations, etc., when this situation comes up in a problem.

Again, if we are looking towards the future, certainly these pupils are going to be better prepared for continuing their work in mathematics than formerly. If anything is worth learning, then surely it is worth retaining. I feel that the only way in which we will be able to cement the various courses in mathematics, is to have a continuous maintenance program extending through all the twelve grades. If we do this, I am confident that our pupils will be better prepared for college or for any work requiring the further use of mathematics than they are at present.

So far, our maintenance program extends only to the end of the tenth year, but provisions are being made to continue it through the entire senior high school. There are many interesting things that have arisen from the use of these drills. Much data is being accumulated and organized relative to various aspects of this program. We hope to have such information available at the close of the present school year.

# In What Respects Should a Child at the End of a Year of Geometry Be Different from What He Was at the Beginning of the Year as a Result of the Training He Receives?*

*By* MARGARET C. BYRNE
*Girls Commercial High School, Brooklyn, New York*

A FORMIDABLE TITLE, is it not? And when you say "a child," what child do you mean?—the child gifted in mathematics, the average, or the low I.Q.? And when you ask how he is to be different, do you mean "measurably different?" And when you append "as a result of the training he receives," do you not really call for a systematic study of a geometry group and a control group?

I shall waive these scientific approaches to the question, and ask myself instead what rational basis I have for my belief that a mastery of the elements of geometry should be required of every secondary pupil. For that is my ardent belief—that the study of geometry pursued under ideal auspices does contribute something essential to the development of a pupil's intelligence, and that no other subject can take its place.

In a brilliant talk last year before the mathematics section of the Experimental Society, Professor Kayser pointed out that there are only two kinds of thinking, that one of these is concerned with reasoning concerning phenomena of the material world and that the other is chiefly concerned with the concatenation of ideas. Geometry epitomizes for the secondary pupil the second of these. I have always liked the subtitle of his Mathematical Philosophy, "A Study in Fate and Freedom"—freedom to choose your postulates but the determining of your conclusions by the inexorable laws of thought. So the first "Good Companion" of the pupil at the end of a year of geometry

* Delivered at the Conference of Teachers of Mathematics in the junior and senior high schools of New York City at George Washington High School on September 16, 1931.

should be some comprehension, call it appreciation if you will, some reverence for these immutable laws of thought.

You may wish to interpolate that science can provide this, in addition to its other excellences. But, of course, the reasoning in science is not subject to the same check. Since the verification of the hypothesis must be made in the material world where new factors may intervene, even the skillful scientist sometimes goes astray, witness the many theories long accepted, later refuted. To expect an inexpert mind, taking its first stumbling steps in rigorous thinking, to avoid these pitfalls is too much. In geometry it is possible to select, from the compact highly organized material, countless problems requiring original thinking that is not beyond the capacity of the beginner. It is only by providing him with sufficient practice in thinking that he develops a feeling of security, learns what constitutes proof. And this is really the second of these "Good Companions"—a recognition of the nature of evidence, of the viciousness of gossip.

In a first term geometry class last term, a girl full of enthusiasm remarked, "What a fine new subject geometry is!" So far as its antiquity was concerned, she might have been talking about television. Perhaps it was poor teaching to wait for this opportunity to bring Euclid into the classroom, but it is a great textbook, is it not, that can stir such enthusiasm after 2,000 years? And if it is true that the greatest service we can perform for our pupils is to bring them into contact with the geniuses of our race, that they may experience "some breathing of a deathless mind," as some one has put it, then our third "Good Companion" is again a spiritual thing—the entering in some degree into the heritage of humanity.

The fourth is esthetic—a feeling for form, for symmetry, for balance, for proportion. It is interesting to note that each of these words which help to define beauty is itself a mathematical word. I should like the year of geometry to flower in an appreciation of the geometric forms in nature as well as in architecture, a notion of dynamic symmetry as well as static, a recognition of the mathematical basis of beauty.

One of the objectives we all accept for our teaching is the stimulating of intellectual curiosity. We accept it; I do not know to what extent we plan to secure it. And surely we include among the desirable outcomes of a secondary education for all the children of all the people an insight into the world in which we live. The laws of this world are fundamentally mathematical, and so are its measurements. How do we measure the distance to a star? Too hard for second year

pupils, and not within the geometry course? Well, how do we measure the height of a building? Book III has always seemed to me a place for pause, for discussion of "Man as the Instrument of his own Measure" (to refer to the famous paintings in the Lincoln School), for simple indirect measurement as a look forward into the theory of Trigonometry, just as Book II is an adventure in relationships. And I have found even the less able pupils stirred by the task of drawing conclusions, building up new sets of facts, from a set of hypothesis, no conclusion being set as the goal. To be entirely honest, this fifth "Good Companion," a lively curiosity, is more often away than present, and I probably have no right to include it in the changes we can expect.

I have left to the last the one I think perhaps the most important, the actual habit of thinking itself. And it is in this connection that we make our most serious mistake. Thinking is hard, a disagreeable task. Yet I have seen pupil after pupil, in a year's patient work, learning to define terms, to set down clearly what is known, to sort out the conclusion that can be drawn, to choose those steps which are helpful in reaching the goal. That is the finest thing we do, but we stop too soon. After all, most of the thinking we need to do is concerned with something other than geometry. Do we ever hear in a geometry class, as we should if I am right, any reference to the way thinking must go on the world outside? It is important to set down what you really know. In the stock speculation of 1929, who cared about the accuracy of his data? It is necessary to take cognizance of necessary conclusions. In political speeches, is there ever remote reference to necessary conclusions? It is necessary to include all factors which may influence the result. In judging our neighbor, are we careful to do that? In certain proofs the indirect method is important. Do we give enough practice in simple examples of this method in non-geometry situations? Professor Upton has pointed out* that the indirect method of reasoning is extremely useful in such situations. This sixth difference then should be the habit of systematic thought in non-school problems.

Here, indeed, are changes to be wrought by a year's study of one subject. Perhaps our list reflects the wish rather than the fact. For these changes envisage the making of a whole new man, new in spirit as in mind.

* Upton, C. B. *The Teaching of Geometry,* Fifth Yearbook of the National Council of Teachers of Mathematics, pp. 102-133. Bureau of Publications, Teachers College, Columbia University, New York City, 1930.

# Irrational Quantities

*By* J. F. Phillipi, *State Teachers College, Buffalo, New York*

Having had in my classes in college algebra hundreds of students coming from a variety of preparatory schools, I have come to the conclusion that the subject of irrationals or radicals is a somewhat difficult subject and that any suggestion as to its presentation to students might be welcomed.

All real numbers are either rational or irrational and both kinds play a very important part in the solution of problems. The rational quantities seem to be handled very well by students but the mystery seems to begin with the irrationals.

Perhaps to add to the confusion is the unfortunate situation caused by the fact that radicals are expressed by the radical sign and also by the fractional exponent, the radical sign being the initial letter of the word radical.

One of the first things a student should know is that the denominator of the fractional exponent indicates the root to be extracted and the numerator indicates the power to which it is to be raised, either of which operations may be done first. Of course, the student will soon find out that in the expression $8^{2/3}$ it is easier to extract the cube root first and then square the result.

Again we find so many examples of the form, "simplify the following radical expression," and the student is at a loss to know when he has arrived at such "simplest form." To answer this query the following has been found to be very helpful. Let us put it this way: When are radicals not in their simplest form? This may be answered by just six cases.

I. $\sqrt{4}=\pm 2$, $\sqrt[3]{8}=2$ that is when the root may be extracted.

II. $\sqrt{8}=\pm 2\sqrt{2}$, $\sqrt[3]{16}=2\sqrt[3]{2}$ when there is a factor in the quantity whose root may be extracted.

III. $\sqrt{\frac{1}{2}}=\pm\frac{1}{2}\sqrt{2}$, $\sqrt[3]{\frac{1}{4}}=\sqrt[3]{2/8}=\frac{1}{2}\sqrt[3]{2}$ when a fraction occurs under the radical the denominator may always be taken out of the radical.

IV. $\sqrt{8}=\sqrt[3]{\sqrt{8}}$ or $\sqrt{\sqrt[3]{8}}=\sqrt{2}$ when the index is composite the root of a root may be taken, each root being a factor of the index.

V. $(a^m)^n = a^m \cdot a^m \cdot a^m \cdots$ to $n$ factors but each $a^m$ has $m$ of the $a$'s and thus equals $a^{mn}$.

VI. $(a^{1/m})^{1/n} = a^{1/mn}$

VII. $a^0 = 1$ that is as the exponent becomes smaller and approaches 0 the power becomes smaller and approaches unity.

VIII. $a^{-b} = 1/a^b$ any quantity with a negative exponent is equal to a reciprocal of the quantity with the same but positive exponent.

Now knowing the meaning of "simplest form" we may proceed to the four fundamental operations with radicals.

In the addition and subtraction of radicals we find the old arithmetical idea prevails, that is, we add like things. Hence, we shall add like radicals. What are like radicals? Naturally they are radicals, which are just alike, as $\sqrt{2}$ is just like $2\sqrt{2}$ but $\sqrt{8}$, by case II becomes $2\sqrt{2}$ so becomes a radical like $\sqrt{2}$ and may be added or subtracted. Thus to add or subtract radicals it is only necessary to submit them to the cases listed above and add or subtract the like ones.

In multiplication we find another condition existing and that is that radicals may be multiplied if their roots are alike regardless of what quantity exists under the radical. This means that all radicals may be multiplied because all radicals may be changed to like roots. To illustrate take $\sqrt{2} \times \sqrt[3]{3}$, unlike radicals, and express them with fractional exponents as $2^{1/2} \cdot 3^{1/3}$. These then may be written $2^{3/6} \cdot 3^{2/6}$ which again changed to the radical form becomes $\sqrt[6]{2^3} \cdot \sqrt[6]{3^2}$ which may be written $\sqrt[6]{2^3 \cdot 3^2}$ or $\sqrt[6]{8 \cdot 9}$ or $\sqrt{72}$. In this manner all radicals having like roots may be multiplied.

The same holds true in division. Thus the dividend and divisor may each be written with fractional exponents and the terms arranged in ascending or descending order when the ordinary method of long division may be followed or dividend and divisor considered the terms of a fraction and the denominator rationalized.

This brings up the process of rationalization of the denominator of a fraction. These fractions fall under three types, the denominator consisting of a single radical, or a binomial surd or a trinomial surd. Taking first the fraction with a denominator consisting of a single radical as $1/\sqrt{2}$, we may always rationalize the denominator by multiplying both terms by something that will cause such denominator to become a power whose root may be extracted. Thus

$$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{\sqrt{4}}=\frac{\sqrt{2}}{2} \text{ or } \frac{1}{\sqrt[3]{3}}=\frac{\sqrt[3]{9}}{\sqrt[3]{27}}=\frac{\sqrt[3]{9}}{3}$$

Again take $1/\sqrt{2}+\sqrt{3}$. Here the denominator is a binomial surd the terms being separated by the plus sign. It will readily be seen that if we multiply both terms of the fraction by $\sqrt{2}-\sqrt{3}$, that is, multiply the sum by the difference it will immediately become the difference of the squares and be rational. If the difference is found in the denominator we multiply by the sum and the denominator will be rational.

Finally, if the denominator is a trinomial surd, we may consider it a binomial by enclosing two terms in parenthesis. Thus $1/(\sqrt{2}+\sqrt{3}+\sqrt{5})$ becomes $1/[(\sqrt{2}+\sqrt{3})+\sqrt{5}]$ and the denominator and numerator may be multiplied by $(\sqrt{2}+\sqrt{3})-\sqrt{5}$. This again amounts to multiplying the sum by the difference. To make clear, we perform the multiplication. $[(\sqrt{2}+\sqrt{3})+\sqrt{5}]$ $[(\sqrt{2}+\sqrt{3})-\sqrt{5}]=(\sqrt{2}+\sqrt{3})^2-5$ or $2+2\sqrt{6}+3-5=2\sqrt{6}$ so we now have the fraction $(\sqrt{2}+\sqrt{3}-\sqrt{5})/2\sqrt{6}$. By now multiplying both numerator and denominator by $\sqrt{6}$ we have $[\sqrt{2}+\sqrt{3}-\sqrt{5}]\sqrt{6}/12$ or $(\sqrt{12}+\sqrt{18}-\sqrt{30})/12$. Thus the fraction now has a rational denominator.

The above brief explanation of computing with radicals having resulted from several years of teaching has cleared the difficulties of so many students in our classes that I submit it with the hope that it may be of some assistance to those who may chance to read it.

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# Applications of Complex Numbers to Geometry

By ALLEN A. SHAW
*University of Arizona, Tucson, Arizona*

*Introduction.* It was with a real pleasure that the present writer read the two excellent articles by Professors L. L. Smail and A. A. Schelkunoff on geometricapplications of thecomplex variable.[1] Both papers are important for the doctrine they expound and for the good training they give the reader in rigorous geometric reasoning on the Argand diagram. While Smail prefers the use of the complex variable in the two-dimensional form, $x+iy$, Schelkunoff employs and recommends the usage of the single variable $z$ to prove the same theorems and obtains very simple and elegant demonstrations.

It is the purpose of this paper to recommend a free use of *both* methods of approach according to convenience, giving preference to the single symbol $z$ whenever possible, as it doubtless affords, in general, shorter and more elegant proofs.[2]

Before writing this article, the present writer read through the illustrations and exercises in a large number of textbooks on college algebra and was amazed not to find even a single example of the type recommended and worked out by Smail and Schelkunoff. It is not, therefore, an exaggeration to say that the two papers (See footnote [1]) quoted above should be regarded as epoch-making in the elementary plane geometry and in the early use of imaginaries.[3] For the power of the use of complex variables, as an instrument of

[1] Smail, L. L.: *Some geometric applications of complex numbers*, American Mathematical Monthly, vol. 36 (1929) pp. 504–511. Schelkunoff, A. A.: *A note on geometrical applications of complex numbers*, American Math. Monthly, vol. 37 (1930) pp. 301–303.

[2] Shaw, A. A.: *Geometric applications of complex numbers*, School Science and Mathematics, vol. 31 (1931) pp. 754–761.

[3] Since these lines were written, Professor Smail has published an excellent text book on college algebra (McGraw-Hill Book Co.) in which the chapter on the complex variable is very well treated in harmony with the two papers cited above. See footnote [1]. This is indeed a refreshing departure from the traditional handling of the subject and can serve as a good model for future text book writers on college algebra.

analysis, is not easy to overrate. In the facility with which these methods deal alike with parallels and perpendiculars, with concyclic and similar figures, with questions of concurrency and collinearity, etc., they can compare not unfavorably with the methods of analytic geometry, and in those questions to which they are specially applicable, the steps necessary for solutions are few in number and quite direct. Consequently the methods of the algebra of imaginaries are too elegant and powerful to be overlooked or even postponed to later years. Indeed, there is no reason why the student should not be encouraged to solve some of his problems by the method of the complex variable in his course on analytical conic sections as well as by the usual Cartesian co-ordinates.[4]

Since the theory of functions, by virtue of its central position in modern analysis, and by virtue of its supreme reign in all parts of the mathematical domain, has now innumerable points of contact with almost every department of science, it is more than necessary for us to teach the freshman student the *elementary* properties of imaginaries without sacrifice of thoroughness. For instance, the student should know such well known notations as $z$, $\bar{z}$, the geometric meanings of $kz$;

$$w = \frac{z_1 - z}{z - z_2}; \quad \frac{z_1 - z_3}{z_2 - z_3} = \frac{z_1' - z_3'}{z_2' - z_3'};$$

etc., otherwise it is meaningless to teach him addition and subtraction by the usual parallelogram-law, or multiplication and division by means of similar triangles, etc., if no further illustrations are to be given to stimulate in him a living interest in the subject. For complex geometry is now playing an increasingly important part in recent mathematical works—pure and applied—hence a complete mastery of the elements of imaginaries by the student is an indispensable prerequisite for the study of function theory and allied subjects. In this connection Dr. Bingley[5] aptly says: "The student and teacher alike should see that imaginaries may be employed as analytic aids at a much earlier stage of mathematical study than is usually the case."

Throughout this paper complex numbers corresponding to points or vertices $A$, $B$, $C$, $D$, etc., of a triangle or quadrilateral, in refer-

[4] Bingley, G. A., *On complex variable in the solution of problems in elementary analytical geometry*. American Mathematical Monthly, vol. 33 (1926) pp. 418–421.

[5] See footnote [4].

ence to some origin $O$, are $z_1$, $z_2$, $z_3$, $z_4$, etc., written in parenthesis after the capital letters as shown in each figure. The conjugate of $z$ is denoted by $\bar{z}$, of $a$ by $\bar{a}$, etc.

In this article we shall make use of the following simple formulas:

*The distance from the point $P_1$ $(z_1)$ to the point $P_2$ $(z_2)$ is given by*

(1) $$P_1P_2 = z_2 - z_1.$$

*The point $P$ $(z)$ which divides in a given ratio $r$ the stroke from $z_1$ to $z_2$ is given by*

(2) $$z = \frac{z_1 + rz_2}{1 + r},$$

where $r$ is real.

When $r = 1$ we have, as a special case, *the mid-point formula*:

(2′) $$z = \tfrac{1}{2}(z_1 + z_2)$$

When $r = 2$, we have *the trisection-point formula*:

(2″) $$z = \tfrac{1}{3}(z_1 + 2z_2)$$

(3) *$kz$ and $z$, $k(z_1 \pm z_2)$ and $(z_1 \pm z_2)$, $kz_1z_2$ and $z_1z_2$, $k\dfrac{z_1}{z_2}$ and $\dfrac{z_1}{z_2}$, all represent parallel vectors in pairs. In each pair the lengths are in the ratio $k$:1.*

(4) (i) $z\bar{z} = r^2 = x^2 + y^2 = |z|^2$;

(ii) *if $z$ lies on the axis of real, $z = \bar{z}$.*

THEOREM 1: *In the $\Delta$ $ABC$, $AD$ and $BE$ are medians intersecting in $O$. Prove that $FGDE$ is a parallelogram.*

PROOF: By formula (2) $O$ is $\frac{1}{3}(z_1+z_2+z_3)$. Then mid-point of $AO$ is

$$\tfrac{1}{2}\left(z_1 + \frac{z_1 + z_2 + z_3}{3}\right) = \tfrac{1}{6}(4z_1 + z_2 + z_3).$$

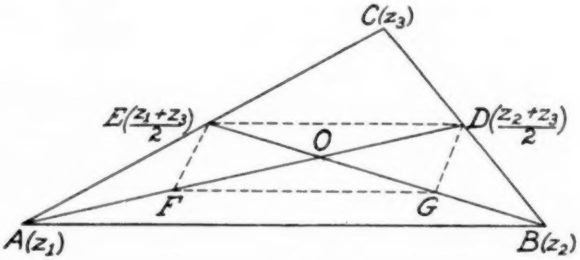


FIG. 1.

Similarly, mid-point of $BO$ is $\frac{1}{6}(4z_1+z_2+z_3)$.

$$\begin{aligned} \text{Hence } FE &= \tfrac{1}{2}(z_1+z_3)-\tfrac{1}{6}(4z_1+z_2+z_3)\\ &= \tfrac{1}{6}(3z_1+3z_3-4z_1-z_2-z_3)\\ &= \tfrac{1}{6}(2z_3-z_1-z_2),\end{aligned}$$

and
$$\begin{aligned} GD &= \tfrac{1}{2}(z_2+z_3)-\tfrac{1}{6}(4z_2+z_1+z_3)\\ &= \tfrac{1}{6}(3z_2+3z_3-4z_2-z_1-z_3)\\ &= \tfrac{1}{6}(2z_3-z_2-z_1).\end{aligned}$$

$\therefore FE$ and $GD$ are equal and parallel, hence the figure $FGDE$ is a parallelogram, by formula (3).

THEOREM 2: *$D$, $E$, $F$ are mid-points of an equilateral triangle $ABC$, prove that the $\Delta$ $DEF$ is equilateral.*

PROOF: Consider the quadrilateral $AFDE$. By formula (1), we have

$$AF=\tfrac{1}{2}(z_2-z_1),$$
$$ED=\tfrac{1}{2}(z_2-z_1).$$

$\therefore$ $AF$ and $ED$ are equal and parallel and the figure is a parallelogram, by formula (3).

Similarly, $FBDE$ and $FDCE$ are parallelograms.

$$\therefore ED=AF=FB=\tfrac{1}{2}s,$$
$$EF=BD=DC=\tfrac{1}{2}s,$$
$$FD=CE=EA=\tfrac{1}{2}s,$$

where $s$ is the side of the $\Delta$ $ABC$.

Hence $\Delta$ $DEF$ is equilateral.

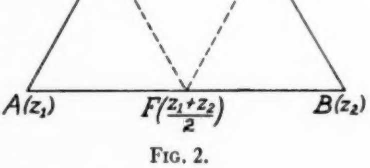


FIG. 2.

THEOREM 3: *If two sides of a quadrilateral are equal and parallel,* (i) *the figure is a parallelogram;* (ii) *the diagonals of this figure bisect each other.*

PROOF: (i) Given $AB$ equal and parallel to $DC$. To prove that $ABCD$ is a parallelogram. Consider the figure as shown. Then, by formula (1) we have

$$z_2 - z_1 = z_3 - z_4 \tag{1}$$

From (1), transposing, we have

$$z_1 - z_4 = z_2 - z_3 \tag{2}$$

(2) shows that $AD$ and $BC$ are also equal and parallel. Hence the theorem is proved.

(ii) Again, from (2), we have

$$\tfrac{1}{2}(z_1 + z_3) = \tfrac{1}{2}(z_2 + z_4) \qquad (3)$$

(3) shows that the mid-point of $AC$ is the same as the mid-point of $BD$. Hence the diagonals of a parallelogram bisect each other.

*Otherwise thus*:

From the figure we have the identity

$$\tfrac{1}{2}(z_1 + z_2) = z_2 + \tfrac{1}{2}(z_1 - z_2)$$

which shows that the diagonals of a parallelogram bisect each other.

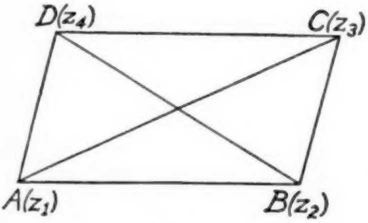


FIG. 3.

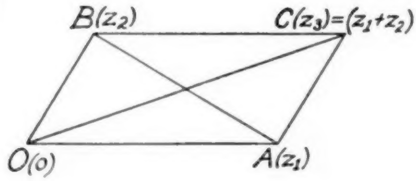


FIG. 4.

THEOREM 4: *The sum of the squares of the diagonals of a quadrilateral is equal to twice the sum of the squares of the lines that join the mid-points of the opposite sides.*

PROOF: From figure 5 we have:

$$AC^2 + BD^2 = (z_3 - z_1)^1 + (z_4 - z_2)^2$$
$$= z_1^2 + z_2^2 + z_3^2 + z_4^2 - 2z_1z_3 - 2z_2z_4. \qquad (1)$$

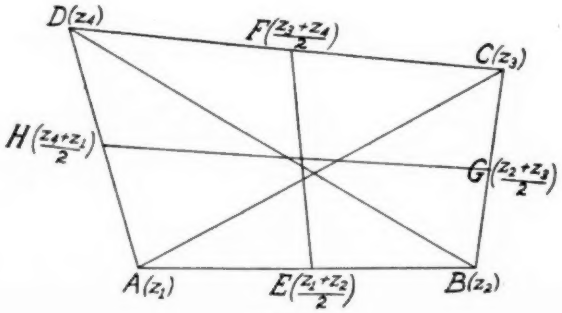


FIG 5.

$$2(EF^2+GH^2)$$
$$=2\left[\left(\frac{z_3+z_4}{2}-\frac{z_1+z_2}{2}\right)^2+\left(\frac{z_1+z_4}{2}-\frac{z_2+z_3}{2}\right)^2\right]$$
$$=\left[\frac{z_3^2+2\,z_3z_4+z_4^2}{4}-\frac{2(z_1z_3+z_1z_4+z_2z_3+z_2z_4)}{4}\right.$$
$$+\frac{z_1^2+2z_1z_2+z_2^2}{4}+\frac{z_1^2+2z_1z_4+z_4^2}{4}+\frac{z_2^2+2z_2z_3+z_3^2}{4}$$
$$\left.-\frac{2(z_1z_2+z_2z_4+z_3z_1+z_3z_4)}{4}\right]$$
$$=\tfrac{1}{2}(z_3^2-2z_3z_4+z_4^2-2z_1z_3-2z_1z_4-2z_2z_3-2z_2z_4$$
$$+z_1^2+2z_1z_2+z_2^2+z_1^2+2z_1z_4+z_4^2-2z_2z_1-2z_2z_4$$
$$-2z_3z_1-2z_3z_4)$$
$$=z_1^2+z_2^2+z_3^2+z_4^2-2z_1z_2-2z_2z_4 \qquad (2)$$

Comparing (1) and (2) the theorem follows.

THEOREM 5: *The sum of the squares of two sides of a triangle is equal to twice the sum of the squares of half the third side and the median which bisects this third side.*

PROOF: We are to prove

$$OA^2+OB^2=2(OM^2+MA^2).$$
$$OA^2+OB^2=z_1^2+z_2^2 \qquad (1)$$

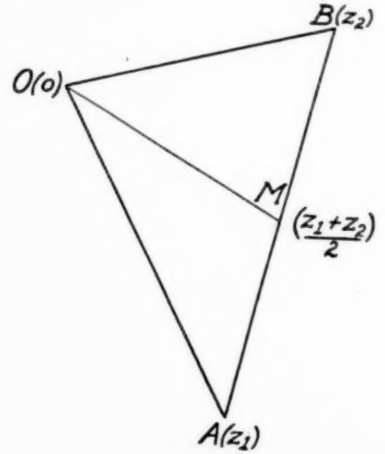


FIG. 6.

$$2(OM^2 + MA^2) = 2\left\{\left(\frac{z_1 + z_2}{2}\right)^2 + \left(\frac{z_1 + z_2}{2} - z_1\right)^2\right\}$$

$$= 2\left\{\left(\frac{z_1 + z_2}{2}\right)^2 + \left(\frac{z_2 - z_1}{2}\right)^2\right\}$$

$$= 2\left\{\frac{z_1^2 + 2z_1z_2 + z_2^2}{4} + \frac{z_2^2 - 2z_1z_2 + z_1^2}{4}\right\}.$$

$$= z_1^2 + z_2^2. \tag{2}$$

Comparing (1) and (2), the theorem is proved.

ALITER. Take $B$ as origin in the figure. We are to prove $AB^2+AC^2=2(BM^2+AM^2)$. Writing $z\bar{z}$ for $AB^2$, $(z-c)(\bar{z}-c)$ for $AC^2$, $(z-c/2)(\bar{z}-c/2)$ for $AM^2$, we have, by formula (4) $z\bar{z}+(z-c)(\bar{z}-c) = 2\cdot c^2/4+2(z-c/2)(\bar{z}-c/2)$ which is identically true and the theorem follows.

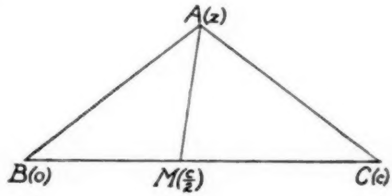


FIG. 7.

COROLLARY 1. *The locus of a point, which moves so that the sum of the squares on its distances from two fixed points is constant, is a circle.*

Let $B(o)$ and $C(c)$ be the fixed points, and let $A(z)$ be any point on the required locus. Then by the condition of the problem, we have, by formula (4)

$$z\bar{z} - \frac{cz}{2} - \frac{c\bar{z}}{2} + \frac{c^2}{2} = k^2,$$

where $k$ is constant and $c$ is real. The last equation may be written

$$\left(z - \frac{c}{2}\right)\left(\bar{z} - \frac{c}{2}\right) = k^2 - \frac{c^2}{4},$$

which is a circle with center $c/2$, and radius $(k^2-c^2/4)^{1/2}$ where $k^2/2=BM^2+AM^2$.

COROLLARY 2. *Find the locus of a point the difference of the squares of whose distances from two fixed points is constant.*

By the condition of the problem we have $AB^2-AC^2=k^2$, where $k$ is constant, i.e., $z\bar{z}-(z-c)(\bar{z}-c)=k^2$ (by formula (4)).

$$\therefore z + \bar{z} = \frac{k^2 - c^2}{c},$$

or

$$2x = \frac{k^2 - c^2}{c}$$

which is a straight line perpendicular to $BC$.

THEOREM 5b. Note that the following is an alternative form of Theorem 5, often quoted as an exercise in textbooks on function theory:

*Show that*

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$$

*and give a geometrical interpretation of this equation.*

PROOF:

$$\begin{aligned}
&|z_1 + z_2|^2 + |z_1 - z_2|^2 \\
&\quad = |x_1 + iy_1 + x_2 + iy_2|^2 + |x_1 + iy_1 - x_2 - iy_2|^2 \\
&\quad = |x_1 + x_2 + i(y_1 + y_2)|^2 + |x_1 - x_2 + i(y_1 - y_2)|^2 \\
&\quad = 2(x_1^2 + y_1^2) + 2(x_2^2 + y_2^2) \\
&\quad = 2|z_1|^2 + 2|z_2|^2
\end{aligned}$$

which proves the theorem. For geometric interpretation see Theorem 5.

THEOREM 6: *If $G$ be the centroid in the $\Delta\ ABC$ and $O$ be any other point, prove that*

(i) $AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$,

(ii) $OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2$.

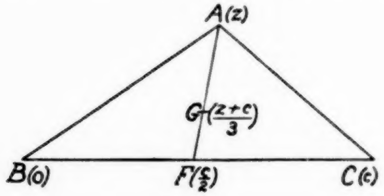


FIG. 8

PROOF: From Fig. 8 we have

$$AB^2 + AC^2 + BC^2$$
$$= z\bar{z} + (z - c)(\bar{z} - c) + c^2$$
$$= 2z\bar{z} - cz - c\bar{z} + 2c^2. \tag{1}$$

$$3(GA^2 + GB^2 + GC^2)$$
$$= 3\left[\frac{4}{9}\left(z - \frac{c}{2}\right)\left(\bar{z} - \frac{c}{2}\right) + w\overline{w} + (w - c)(\overline{w} - c)\right]$$
$$= \frac{4}{3}z\bar{z} - \frac{2}{3}cz - \frac{2}{3}c\bar{z} + \frac{c^2}{3} + \frac{2}{3}z\bar{z} - \frac{c}{3}z - \frac{c}{3}\bar{z} + \frac{5}{3}c^2$$
$$= 2z\bar{z} - cz - c\bar{z} + 2c^2 \tag{2}$$

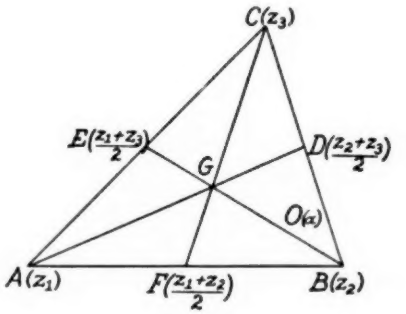


FIG. 8a.

where $\omega$ is the affix of $G$ and is equal to $z+c/3$. Comparing (1) and (2) the theorem follows.*

(ii) $OA^2 + OB^2 + OC^2$

$$= (z_1 - \alpha)^2 + (z_2 - \alpha)^2 + (z_3 - \alpha)^2$$
$$= z_1^2 + z_2^2 + z_3^2 + 3\alpha^2 - 2\alpha z_1 - 2\alpha z_2 - 2\alpha z_3. \tag{1}$$

$$GA^2 + GB^2 + GC^2 + 3GO^2$$
$$= (z_1 - \omega)^2 + (z_2 - \omega)^2(z_3 - \omega)^2 + 3(\alpha - \omega)^2, \quad \text{where}$$

$$\omega = \frac{z_1 + z_2 + z_3}{3},$$

* For another proof of this, see A. A. Shaw, "*Geometric Applications of Complex Numbers*," School Science and Mathematics, vol. 31, p. 760.

$$= z_1^2 + z_2^2 + z_3^2 - 2\omega(z_1 + z_2 + z_3) + 6\omega^2 + 3\alpha^2 - 6\alpha\omega$$

$$= z_1^2 + z_2^2 + z_3^2 + 3\alpha^2 - 2\omega \cdot 3\omega + 6\omega^2 - 6\alpha\omega.$$

$$= z_1^2 + z_2^2 + z_3^2 + 3\alpha^2 - 2\alpha z_1 - 2\alpha z_2 - 2\alpha z_3. \tag{2}$$

Comparing (1) and (2) the theorem is proved.

THEOREM 7: *The median of a trapezoid bisects both diagonals.*

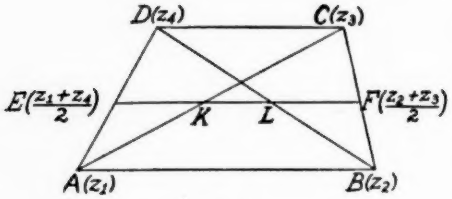


FIG. 9.

PROOF: In the figure $AB$ is parallel to $CD$, $EF$ is the median, and $K$, $L$ are the mid-points of the diagonals. Since, by formula (1), $EK = \frac{1}{2}(z_1+z_3) - \frac{1}{2}(z_1+z_4) = \frac{1}{2}(z_3 - z_4)$, then $EK$ is parallel to $DC$ and, therefore, to $AB$; and since $FK = \frac{1}{2}(z_2 - z_1)$, then $KF$ is parallel to $AB$ and, therefore, to $DC$. Hence $K$ lies on $EF$.

Similarly, since $EL = \frac{1}{2}(z_2 - z_1)$, then $EL$ is parallel to $AB$ and, therefore, to $DC$; and since $LF = \frac{1}{2}(z_3 - z_4)$, then $LF$ is parallel to $DC$ and, therefore, to $AB$. Hence $L$ lies on $EF$ and the theorem is proved.

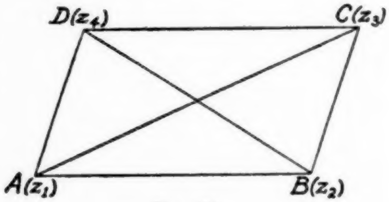


FIG. 10.

THEOREM 8: *The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.*

PROOF: Since the figure is a parallelogram, we have by formula (1)

$$z_2 - z_1 = z_3 - z_4,$$

$$\therefore z_2 = z_1 + z_3 - z_4 \tag{1}$$

from Fig. 10, by formula (1), we have

$$AB^2 + DC^2 + BC^2 + AD^2$$

$$= 2(DC^2 + AD^2)$$

$$= 2\{(z_3 - z_4)^2 + (z_4 - z_1)^2\}$$

$$= 2z_1^2 + 2z_3^2 + 4z_4^2 - 4z_1z_4 - 4z_3z_4. \tag{2}$$

$$AC^2 + BD^2$$
$$= (z_3 - z_1)^2 + (z_4 - z_2)^2$$
$$= (z_3 - z_1)^2 + (2z_4 - z_1 - z_3)^2,$$

substituting from (1),

$$= 2z_1^2 + 2z_3^2 + 4z_4^2 - 4z_1z_4 - 4z_3z_4. \qquad (3)$$

Comparing (2) and (3), the theorem is proved.

THEOREM 9: *If the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on the diagonals, the quadrilateral is a parallelogram.*

PROOF: Given that

$$(z_3 - z_2)^2 + (z_2 - z_1)^2 + (z_3 - z_4)^2 + (z_4 - z_1)^2$$
$$= (z_4 - z_2)^2 + (z_3 - z_1)^2. \qquad (1)$$

To prove that the figure is a parallelogram. From (1) we have

$$(z_3 - z_4)^2 = (z_4 - z_2)^2 + (z_3 - z_1)^2 - (z_2 - z_1)^2 - (z_4 - z_1)^2$$
$$- (z_3 - z_2)^2$$
$$= 2(z_2z_3 - z_1z_3 + z_1z_4 - z_2z_4) - z_2^2 - z_1^2 + 2z_1z_2$$
$$= 2(z_3 - z_4)(z_2 - z_1) - (z_2 - z_1)^2.$$
$$\therefore \{(z_3 - z_4) - (z_2 - z_1)\}^2 = 0,$$
$$\therefore (z_3 - z_4) - (z_2 - z_1) = 0,$$

or $z_3 - z_4 = z_2 - z_1$. Hence the figure is a parallelogram, two sides being equal and parallel.

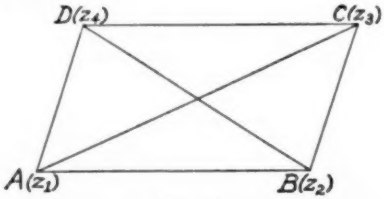


FIG. 11.

THEOREM 10: *In any quadrilateral, two of whose opposite sides are parallel, the sum of the squares on the diagonals is equal to the sum of the squares on its non-parallel sides, together with twice the rectangle contained by the parallel sides.*

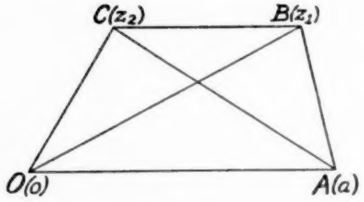


FIG. 12.

PROOF: In the figure $OA$ is parallel to $CB$ and $O$ is taken as origin. We are to prove that

$$OB^2 + AC^2 = OC^2 + AB^2 + 2 \cdot OA \cdot CB.$$

Writing $z_1\bar{z}_1$ for $OB^2$, $z_2\bar{z}_2$ for $OC^2$, $(z_1-a)$ $(\bar{z}_1-a)$ for $AB^2$, etc., we have, by formula (4)

$$\begin{aligned} OB^2 + AC^2 &= z_1\bar{z}_1 + (z_2 - a)(\bar{z}_2 - a) \\ &= z_1\bar{z}_1 + z_2\bar{z}_2 - az_2 - a\bar{z}_2 + a^2 \\ &= z_1\bar{z}_1 + z_2\bar{z}_2 + a(z_2 - \bar{z}_2) - 2az_2 + a^2 \\ &= z_1\bar{z}_1 + z_2\bar{z}_2 + a(z_1 - \bar{z}_1) - 2az_2 + a^2, \qquad (1) \end{aligned}$$

for $z_1-\bar{z}_1=z_2-\bar{z}_2$, $CB$ being parallel to $OA$.

$$\begin{aligned} &OC^2 + AB^2 + 2 \cdot OA \cdot CB \\ &\quad = z_2\bar{z}_2 + (z_1 - a)(\bar{z}_1 - a) + 2a(z_1 - z_2) \\ &\quad = z_2\bar{z}_2 + z_1\bar{z}_1 - az_1 - a\bar{z}_1 + a^2 + 2az_1 - 2az_2 \\ &\quad = z_2\bar{z}_2 + z_1\bar{z}_1 + az_1 - a\bar{z}_1 + a^2 - 2az_2 \\ &\quad = z_1\bar{z}_1 + z_2\bar{z}_2 + a(z_1 - \bar{z}_1) - 2az_2 + a^2. \qquad (2) \end{aligned}$$

Comparing (1) and (2) the theorem follows.

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# Grading Entirely by Tests

*By* P. Stroup, *West High School, Cleveland, Ohio*

I was much interested in the article by Nelson A. Jackson in the January number of The Mathematics Teacher on "Home work in first year Algebra." I was not surprised but rather much comforted by statistical confirmation of my suspicion that the inspection of home work is a poor basis for grades. There is a better use for home-work and also a better basis for grades. The students correct their own home work or at least are given an opportunity to profit by their own mistakes. Every day the problems are shown to the students correctly solved but after they have had an opportunity to solve them themselves. The usual program is to read the answers to the home work problems and then have some student who has a certain problem solved correctly put it on the board. All are given opportunity to criticize and ask questions. If this gives too much leisure for some, the lesson for the next day is given out and those who are ready go ahead with it. The teacher is constantly on call for any one who wants help; sometimes interrupting the whole group to save time by making a general explanation.

The tests come about once a week and were planned to cover units of the work as fractions, quadratics, logarithms. After the test papers had been marked and returned and the mistakes shown by correct work on the board, the pupils that did not get a mark that satisfied them take another test on that unit outside of class time securing more help from the teacher or from any other source. For each retaking of the test one was subtracted from the month's grade. These tests were made so as to sufficiently cover the subject matter in order that the "retakers" had little advantage over those taking the original test. The ambitious students who retook the test in order to raise the test mark a point or two really earned the addition to their monthly grade by the additional algebraic experience they got by taking the test.

The plan was discussed in each class at the close of the semester and no real objection was found even on the part of those students who failed to take advantage of the opportunity to retest. The plan places the responsibility largely on the pupil and therein is its weak-

ness if such it should be called. I felt sure that many students had said to themselves when the question came up whether to do their homework or something else, "It isn't marked anyway." This is in a way a point in favor of the plan. It makes the student's home work more flexible in amount and a conscientious student can when pressed by home exigencies or other home work neglect his algebra for one day. There is no doubt that some of the weaker characters make a habit of this but the tests if well constructed will catch them. More serious is the opportunity for a bright student to get by merely paying attention in class.

Also I had a feeling that when a test loomed some students instead of preparing said to themselves, "If I don't get it the first time I can retake it." The one-point penalty was not heavy enough to deter them. To break up this habit I announced that the retake tests would be made harder than the first one and did my best to make it so.

A fine example of the very poor habits that some students have was shown in the persistence with which some students copied problems from the board after the positive announcement that the home work was only used to see what mistakes the class was making and where help was most needed.

This is by far the most satisfactory plan I have ever tried for conducting algebra classes especially those for upper grade students. Most of them are ready to accept the responsibility offered them and the teacher can soon pick out those who need his urging though I doubt that it will do as much good as to let the student meet the consequences of his own conduct.

## National Council Membership Increasing

In spite of the depression the National Council of Teachers of Mathematics has increased its membership during the past year.

Substantial gains have been made in subscribers to The Mathematics Teacher in most states of the Union. West Virginia is by all odds the banner state, having doubled their membership. The states that have lost ground are the following: Arizona, Delaware, Florida, Idaho, Indiana, Michigan, Mississippi, Nevada, North Carolina, Oklahoma, Oregon, South Carolina, Utah, and Virginia.

The Editor

# Blaise Pascal

**Born at Clermont-Ferrand, June 17, 1623**

**Died in Paris, August 19, 1662**

BLAISE PASCAL is known as a philosopher, a physicist, an inventor, and a mathematician, but his varied activities were so closely allied that it is difficult to discuss any one without discussing the others also. It is difficult, too, to speak of the circumstances of his life without constant reference to his father who was a lawyer, and to his two sisters Mme. Perier and Jacqueline.

It was to further the education of his children that Etienne Pascal took his family from Clermont to Paris. The connection with their old home was never broken and the older sister eventually married and settled there. The story is told that the young Pascal was trained in the classics rather than in mathematics and that following a chance suggestion, he invented much of the first book of Euclid's *Elements* by himself. This last is probably without foundation.

In Paris, the literary gifts of the younger sister attracted favorable attention from influential people at the court and seem to have been the means of securing a pardon from Richelieu for some political offense of the father's and won for him also a government appointment at Rouen. Before leaving Paris, however, the younger Pascal had distinguished himself by presenting an essay on conics at one of Father Mersenne's conferences.[1] Mersenne wrote enthusiastically to Descartes saying that Blaise Pascal had overtopped the Greeks, but Descartes replied coldly that this was no matter to boast of. The essay on conics[2] contained the theorem now often called the Mystic Hexagram: If a hexagon is inscribed in a conic, the intersections of the pairs of opposite sides are collinear. From this theorem, Pascal later claimed that he could deduce several hundred corollaries.

The principal part of Etienne Pascal's work in Rouen was to straighten out the tax books. To help in the tedious computations which this involved, Blaise Pascal invented the first adding machine (1640).[3] It was hoped that this machine would make the family for-

[1] See *The Mathematics Teacher*, October 1931, p. 369.

[2] For a translation, see the *Source Book in Mathematics* edited by David Eugene Smith, New York, 1929, p. 326.

[3] Ibid., p. 165. Pictures of a copy of Pascal's machine are given in *A Short*

tune but the inevitable imitations were made and the financial profit does not seem to have been large. At a later date, Pascal undertook a greater venture, the establishing of the first omnibus lines in Paris.

In 1646, Mersenne wrote to Pascal about the experiments recently made with the barometer by Galileo's pupil Torricelli. Better equipment for duplicating this could be made by the glass blowers of Rouen than by those of Paris, and Pascal repeated Torricelli's experiments and directed his brother-in-law in the study of the variation in pressure over a range in elevation of some 3000 ft. on Puy-le-Dôme near Clermont. It should be remembered that at this time, scientists were not wholly convinced as to the existence of a vacuum.

Pascal's other investigations at this period involved the laws of fluid pressure and the invention of the hydraulic press.

Pascal's residence in Rouen was marked by the conversion of the younger members of the family to Jansenism. The Jansenists have been described as being a Puritan group within the Roman Catholic Church. Jacqueline Pascal entered the Jansenist sisterhood of Port Royal de Paris in 1652 soon after the death of her father. Blaise Pascal spent considerable time in retreat at the Jansenist institution of Port Royal des Champs near Versailles. Here is being assembled a collection of things associated with Pascal, among them a copy of the wheelbarrow which he invented and an arrangement of pulleys so made that a small boy could raise a barrel-full of water from a well. Another reminder of Pascal's association with the Jansenists is his *Provincial Letters* written under the nom de plume of Louis de Montalte defending the group against the attacks of their enemies.

After the Rouen experiences, Pascal again lived in Paris, keeping in touch with Mersenne's group and being tutored in the ways of society by the Chevalier de Méré who has been described as an "arbiter elegentiarum." It was fashionable to dabble in mathematics and science at this time and Méré who was more able than most was the occasion of the correspondence between Fermat and Pascal regarding a gambling problem. This correspondence as has been previously noted led eventually to the study of probability. In this Pascal made use of the array of figures which give the coefficients of the binomial theorem. This had been known long before, but Pascal's extensive use

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*History of Mathematics*, by Vera Sanford, Boston, 1930, p. 351. A facsimile of the essay on conics appears on p. 286.

of it and his treatise on the Arithmetic Triangle[4] have given it his name.

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| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
| 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | | |
| 1 | 4 | 10 | 20 | 35 | 56 | 84 | | | |
| 1 | 5 | 15 | 35 | 70 | 126 | | | | |
| 1 | 6 | 21 | 56 | 126 | | | | | |
| 1 | 7 | 28 | 84 | | | | | | |
| 1 | 8 | 36 | | | | | | | |
| 1 | 9 | | | | | | | | |
| 1 | | | | | | | | | |

THE PASCAL TRIANGLE

Pascal's second conversion to Jansenism took place in 1654. After this, he was occupied with philosophy and paid little attention to mathematics and physics. In 1658, however, having worn himself out in this other writing, he turned to the study of the cycloid and found in it the relief he desired. His friends are said to have urged his utilizing his findings to show the skillfulness of his reasoning in geometry, the inference being that his reasoning in theology was equally rigorous. Pascal, accordingly, recast his old nom de plume and as Amos Dettonville offered a prize for the best solution of certain problems regarding the cycloid. The English contestants disputed about the time limits for the English were still using the Julian calendar while France was using the Gregorian one—thus the English contributions were due some days before the close of the competition according to the English calendar. Pascal and his advisers kept to the original limits with the interesting suggestion that some date had to be set otherwise years after the prize was awarded, contributions might arrive from out of the way villages in China. John Wallis's solution was rejected. Sir Christopher Wren's was spoken of with some approval. No one was given the prize, but awkward controversies as to plagiarism and priority followed.

Pascal had always been in frail health. By 1660 his condition was made much worse by the persecution of the Jansenists by the Jesuits and the court party. Pascal as usual came to the defense of the Jansenists but he was unable to prevent the measures he opposed. His sister Jacqueline died in 1661, Pascal in the following year.

VERA SANFORD

[4] *Source Book in Mathematics*, p. 67.

# John Wesley Young

**November 17, 1879-February 17, 1932**

TEACHERS of mathematics in secondary schools in this country have been fortunate in having the help and leadership of certain few scholars of pure mathematics who, like Felix Klein in Germany, have been keenly interested in the problems of the elementary field. These men are rare, for such work requires unusual gifts. Of the group, the late Prof. J. W. Young was an outstanding member.

He was born at Columbus, Ohio, studied in this country and in Europe, received his bachelor's degree from Ohio State in 1899 and his master's and doctor's degrees from Cornell in 1901 and 1904. He taught at Northwestern University, the University of Chicago, Princeton, the University of Illinois, the University of Kansas, and Dartmouth College.

His early publications, notably his *Projective Geometry* (1910) in collaboration with Professor Veblen, were in the field of pure mathematics. His *Lectures on the Fundamental Concepts of Algebra and Geometry* (1911) indicate the trend of his work in later years. In the preface of the latter work he remarked, that "The results of recent work on the logical foundations are of vital interest alike to the teachers of mathematics in our secondary schools and colleges and to philosophers and logicians. I hope that both these classes will welcome a concise statement of some of the more fundamental of these results and an elementary exposition, omitting all involved details, of the point of view which governs all present work in the foundations." One part of this hope has been amply realized in the use of this volume in numerous courses for teachers.

From 1913 to 1915 Professor Young served as chief examiner in geometry for the College Entrance Examination Board. In 1916 the National Committee on Mathematical Requirements was organized under the auspices of the Mathematical Association of America "in response to an insistent demand that national expression be given the various movements toward reform in the teaching of mathematics which had gained more or less headway through the activities of various local organizations throughout the country."[1] Professor Young

[1] J. W. Young, "The Work of the National Committee on Mathematical Requirements," THE MATHEMATICS TEACHER, XIV (January, 1921), p. 5.

was appointed chairman of this committee, the active work of the group continuing over a period of six years and culminating in the publication of the *Reorganization of Mathematics in Secondary Education* (1923), better known as the "Report of the National Committee." In 1919-20, the chairman was enabled to devote his full time to the work of the committee and it was during this year that many of us first became personally acquainted with him. His able presentation of the work of the committee and his exposition of its purposes undoubtedly had much to do with the organization of the National Council of Teachers of Mathematics, while his courteous attention to the often trivial suggestions of his auditors inspired us to feel that we too might accomplish much, and we felt that the final report was in part our work. It should be noted also that Professor Young served on the editorial board of THE MATHEMATICS TEACHER from the time when it was taken over by the Council until 1928.

In 1929, as president of the Mathematical Association of America, Professor Young took the first steps in the organization of the joint committee on geometry whose report appeared in THE MATHEMATICS TEACHER for May, 1931. He was also a member of the American Subcommittee of the International Commission on the Teaching of Mathematics whose investigations are now in process.

His publications include several textbooks: *Plane Geometry* and *Solid Geometry* with A. J. Schwartz; *Elementary Mathematical Analysis* (1917) with F. M. Morgan, a pioneer text in the reorganization of freshman college courses; a *Plane Trigonometry* with the same collaborator; and the Carus Monograph *Projective Geometry* (1930). He was also on the editorial staff for the *Bulletin* of the American Mathematical Society for upwards of twenty years, and after 1924 served as mathematical editor for the Houghton Mifflin Company.

Those of us who were so fortunate as to know him personally remember the delightful sense of humor that enlivened even the drudgery of proofreading as when he suggested the student's idea that the method of exhaustions was so named from its effect on the user. We remember the quick turn of phrase that would draw attention to the remark of some novice speaking on the same program, rather than claiming the point as his own though the chance was strong that the novice had taken it from him in the first place. When his election as president of the Association was announced, he told someone who congratulated him that he had just sent a wire to the defeated candidate "Condolences and Congratulations," but remarked that if the

results had been reversed, the telegram would have read in the reverse order.

Perhaps no better comment could be made on his work than to quote the closing paragraph of his recent address on "The Functions of the Mathematical Association." No one who knew Professor Young would fancy that he would think of his own career as illustrating his remarks. Yet no one who knew him well can fail to see the parallel.

It is probably impossible to determine the relative value of a guard and a back on a football team. It is probably even more difficult to determine the relative value, to the mathematical organism as a whole, of the research man and the man who labors to improve the conditions which make research possible and which give it significance. Both are essential. . . . There is important work for all of us. The sin of the mathematician is not that he doesn't do research, the sin is idleness, when there is work to be done. If there be sinners in my audience I would urge them to sin no more. If your interest is in research, do that; if you are of a philosophical temperament, cultivate the gardens of criticism, evaluation, and interpretation; if your interest is historical, do your plowing in the field of history; if you have the insight to see simplicity in apparent complexity, cultivate the field of advanced mathematics from the elementary point of view; if you have the gift of popular exposition, develop your abilities in that direction; if you have executive and organizing ability, place that ability at the disposal of your organization. Whatever your abilities, there is work for you to do—for the greater glory of mathematics. And this, I think, is the nearest I have ever come to preaching a sermon.[3]

VERA SANFORD

[3] *American Mathematical Monthly,* XXXIX (January, 1932), p. 15.

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# Eighth Yearbook

The eighth yearbook of the National Council of Teachers of mathematics will be devoted to "The Mathematics of Secondary Schools of the United States." This will not be a record of the present status altogether, but rather a presentation of some of the more progressive practices. If any of the Council members know of significant studies or experiments, please let us know about them.

THE EDITOR

# NEWS NOTES

The following poem was written by Ray Dorsey, a student in an intermediate algebra class taught by Miss Hattie Selover in Shaker Heights, Cleveland, Ohio.

*To Mathematics*

In imaginations hidden realm,
  By the banks of mortal mind,
There's a castle strongly standing,
  Which the keen alone can find.
There sequestered midst the darkness,
  With, as guardsmen, owls wise,
Stands this castle, Euclid's palace
  Watchèd by celestial eyes.
And for all who enter herein
  There's a treasure rare as gold
There's precision, which is godly,
  There is speed, whose worth's untold.
In this mighty magic region,
  In this realm of the mind,
Archimedes, Euclid, Newton,
  Much did seek, yet more did find.
Tho' for aeons, men have gone there,
  Of this treasure to partake,
It, like Jove's o'erflowing goblet.
  Still is full, our thirst to slake.

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Inspired by a suggestion in the reference to the Lewis Carroll Celebration in the January issue of THE MATHEMATICS TEACHER Miss Elizabeth Sadley of the Northrop Collegiate School of Minneapolis with the help of her ninth and tenth grade classes in mathematics put on the following program on January 27, 1932:

I. Biographical Sketch.
II. Alice in Wonderland (Alice Liddell, the real Alice).
III. Lewis Carroll's love for children, as shown by his letters.
IV. Humorous References to Mathematics, in the writings of Lewis Carroll.
V. The exhibit of Carrolliana at Columbia University, in April, 1932.

Sources of material used:

I. Life and Letters of Lewis Carroll. Stuart Dodgeson Collingwood.
II. Life of Lewis Carroll. Belle Moses.
III. New York Times Magazine. Jan. 23, 1932.
IV. The Mathematics Teacher. Jan. 24, 1932.
V. The Scholastic Magazine. Jan. 24, 1924.

Miss Sadley says that the program not only created a favorable impression with all pupils beginning with the little third graders and extending through the high school, but that it was thoroughly enjoyed by the faculty and visitors as well.

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The Mathematics Section of the Association of Teachers of Mathematics of the Middle States and Maryland met at 10:30 A.M. in Haddon Hall at Atlantic City on November 18, 1931. President, Mr. Donald E. MacCormick, William Penn Charter School, Philadelphia; Secretary, Miss Norma Sleight, Shippen School, Lancaster, Pennsylvania.

The following program was given:

1. Business Meeting with election of officers.

2. Application of Groups to Some Famous Problems in Geometry, Dr. Howard Hawkes Mitchell, University of Pennsylvania.

3. A One Year Course in Plane and Solid Geometry—Is It Desirable or

Feasible? Discussion to be led by Dr. Fletcher Durrell, Vice-president.

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The following program was rendered at the College Mathematics section of the Oklahoma Education Association on February 5, 1932, at Oklahoma City:

"Why the High School Teacher of Mathematics Should Have a Knowledge of the Calculus," Prof. G. Emory Meador, Oklahoma City University, Oklahoma City, Oklahoma.

"Vectoral Analysis Applied to Projective Geometry," Dr. E. F. Allen, Oklahoma A. & M. College, Stillwater, Oklahoma.

"Mathematics and Beauty," Dr. Nathan Altshiller-Court, University of Oklahoma, Norman, Oklahoma.

"Shall the Teachers Colleges Teach Arithmetic?" Dr. Clarence McCormick, Southwestern State Teachers College, Alva, Oklahoma.

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One hundred and fifty people were present at the February meeting of Section 19 (Mathematics) of the New York Society for the Experimental Study of Education of which Prof. W. D. Reeve of Teachers College, Columbia University, is chairman. Prof. Earl R. Hedrick, chairman of the American Committee of the International Commission on the Teaching of Mathematics, and professor of mathematics at the University of California at Los Angeles, spoke on "Formalism in Mathematics Teaching," and Mr. Joseph P. McCormack of Theodore Roosevelt High School, gave a report on the meeting of the National Council of Teachers of Mathematics held in Washington, D.C., on February 19 and 20.

On March 26 Mr. C. E. Trueblood of Indianapolis spoke on "Teaching Mathematics in Large Size Classes" and Mr. J. T. Johnson of the Chicago Normal School spoke on "A Teaching Technique that Administers to Individual Differences."

On April 30, Prof. W. R. Longley of Yale University will be the guest of the club and will speak on "The Function Idea in Algebra."

These dinner meetings are held at 6:30 P.M. at the Men's Faculty Club, Columbia University, 117th Street and Morningside Drive. The addresses are given at 7:30 P.M. Everyone is welcome who is interested.

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The forty-fourth regular meeting of the Association of Mathematics Teachers of New Jersey was held in Fine Hall at Princeton University on Saturday, March 5, 1932.

Morning Session

"A Problem in Analysis Situs," by Dr. Leo Zippen, Princeton University.

"Industrial Mathematics in Camden Junior High Schools," by Miss Marie Lukens, Camden Senior High School.

"Duplex Algebra," by Mr. Harrison E. Webb, Market Street High School, Newark.

Afternoon Session

"Recent Developments in Astronomy," by Dr. Henry Norris Russell, Princeton University.

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Fine Memorial Hall is situated on the southern side of Princeton campus, adjoining Palmer Physical laboratory, and overlooking the tennis courts.

Fine Hall was erected last year through the generosity of Miss Gwenthalyn Jones of Chicago, and her uncle, the late Thomas D. Jones, Princeton, '76, in memory of Dean Henry Burchard Fine, for many years the head of the department of mathematics of the university, and during his later years dean of the faculty of Arts and Sciences.

This Association will remember with appreciation the interest of Dean Fine

in its organization as shown by his membership on the Council from its beginning and his acceptance of the presidency for the second year of its existence, as well as an active part in the arrangement of the programs of many of the meetings.

Quoting from Dean Eisenhart's description of Fine Hall in a recent number of the *Princeton Alumni Weekly,* this summary of the arrangement of the building and its purposes will be of interest to the association:

"The building is intended to be a center for the study of mathematics and mathematical physics. It provides studies for members of the staff and similar facilities for advanced graduate students. There are several lecture and seminar rooms and a reading room for the use of upperclassmen enrolled in the department of mathematics. One of the outstanding features of the building is the library of the departments of mathematics and physics which occupies the whole third floor and includes four conference rooms, as well as alcoves in which the advanced students can carry on their studies. On the second floor there is a large conference room for meetings of the professors, and another room, 'the common room' which will be a general meeting place for the members of the staff and the students. The portrait of Dean Fine, which has just been painted by Ernest Ipsen, will be the central feature of the latter room.

"This building will provide for those interested in mathematics and mathematical physics opportunities for a group life similar to that enjoyed by our departments of science in their laboratories."

OFFICERS OF THE SOCIETY

Roscoe P. Conkling.........*President*
Central High School, Newark, N.J.

Virgil S. Mallory.......*Vice-President*
State Teachers College, Montclair, N.J.

Andrew S. Hegeman..............
..............*Secretary-Treasurer*
Central High School, Newark, N.J.

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The Mathematics Club of the Brooklyn Technical High School under the leadership of Morris Cohen, chairman of the department of mathematics, has just published No. 1 of Volume I of a very interesting paper called the "Brooklyn Technical High School Mathematics Student." It should be of interest to other clubs over the country.

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Mr. Henry H. Shanholt, chairman of mathematics at the Abraham Lincoln High School, will conduct a course in mathematics at the Adirondack Summer School for Teachers, located at Raquette Lake, New York. The course will offer intensive preparation for teachers who plan to take examinations to teach in the junior and senior high schools of New York City, also for the first assistants' test soon to be scheduled. The course will include a discussion of the best and most progressive methods of teaching mathematics.

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On February 28 there died at Providence, Rhode Island, Dr. Arnold Buffam Chace, chancellor (president of the corporation) of Brown University. His edition of the Rhind Mathematical Papyrus is probably the most elaborate publication ever made of any ancient mathematical manuscript, and is certainly one of the most scholarly. For more than a half century he was a member of the Brown Corporation, and during most of this time he held either the office of treasurer or that of chancellor. He was educated at Brown and did advanced work in chemistry in Europe.

# ◆ NEW BOOKS ◆

*The Queen of the Sciences.* By E. T. Bell, of The California Institute of Technology. Baltimore, Williams and Wilkins, 1931. pp. iv + 138. Price $1.00.

It is a trite and rather patronizing statement that a certain book should be read by every high school teacher. If it were to be used at all, however, it might be used in connection with Professor Bell's charming little work, which came out a few months ago. It is a combination of historical information and material to show the nature of the various branches of elementary mathematics, with some excursions into such fields as complex numbers, transformations, groups, algebraic numbers, transcendental numbers, and the infinite in mathematics. It begins with a treatment of the purposes of mathematics, and ends with a reference to some of the prominent theories of the last quarter of a century. It allows the reader to find out without undue difficulty the nature and bases of the postulates of mathematics, the development of the underlying rules of the science, the significance of invariants and projections, and the nature of geometry as considered by mathematics of the present day.

Naturally the work is not an exhaustive treatise upon any of the branches of mathematics, but it is an excellent summary of some of the great features which have been developed in the last two centuries, and particularly since about 1850.

Professor Bell's skill as a novelist is carried over to enliven his writing upon a somewhat abstruse subject. It would be difficult to find a better way of spending a dollar than that which leads to the buying of this book.

David Eugene Smith

*The Algebra of Omar Khayyam.* By Daoud Kasir, The Bureau of Publications, Teachers College, Columbia University, 1932. pp. iv + 126. Price $2.00.

This book presents, in two sections, the scientific work of Omar Khayyam in the field of algebra, based on an Arabic manuscript taken from the library of Prof. David Eugene Smith The first section, which is the translator's introduction, falls under four headings.

1. *Omar Khayyam in the East.*—A brief account of his life history, training, achievements in other fields of science, poetry and literature. Brief mention is also made of the influence such works had upon contemporary and later writers in the East as indicated by several critical and favorable comments found in Arabic sources.

2. *Omar Khayyam's algebra in the West.*—This shows the manner in which Omar Khayyam's algebra was introduced into the West through a few copies of his manuscripts found in European libraries, particularly that of the library of Leyden. These manuscripts are described and a somewhat extended statement is made describing the significance of Professor Smith's manuscript. In addition a general review is given of all comments on the work of Omar Khayyam from the time the first manu-

script was found, in 1742, until the present time.

*3. A survey of the development of algebra before the time of Omar Khayyam.*—In this section a general survey of Greek and Arabic achievements in the field of algebra is given, particularly those parts that influenced the work of the author. An account is also given of the particular problems that were of greatest interest to Greek as well as to subsequent Arabic mathematicians, and which are responsible for the more systematic theorems of Omar Khayyam. In addition, reference, is made to the influence of Hindu work upon the achievements of Arabic writers and in turn upon the work of Omar. Finally, the distinct difference between Greek and Arabic mathematics is pointed out, together with the factors that led the Arabic method of thinking to diverge from that of the Greeks. The Greeks developed an abstract and deductive theory of mathematics, while the Arabs evolved an analytical interpretation and practical application of Greek theories.

*4. Omar Khayyam's method and contribution.*—Under this heading Omar's text is described in some detail. A detailed analysis of his method and conclusions are also given. Special reference is made both of his failure to arrive at certain logical conclusions, and to the distinct contribution he made to the science of algebra. Furthermore, particular attention has been given to the problem of tracing the origin of some of Omar's conclusions back to the work of previous Greek and Arabic mathematicians. Finally, a critical discussion of the classifications of equations that he proposed to demonstrate is given from the standpoint of modern algebraic theory.

The second section of the present work includes the translation of Omar's text supplemented by explanatory and bibliographical footnotes. The text is conveniently divided into ten chapters dealing with the different topics proposed by the author.

In his introduction Omar points out the reason for taking upon himself the task of writing such a work. Also he makes special mention of the difficulties that he encountered when preparing his work. He relates also the progress of mathematics among Arabic scholars and adds a critical analysis of their work. He defines the fundamental notions of algebra and sets forth what he considers to be the prerequisite knowledge of mathematics necessary for comprehension of his own work. He provides a table of equations which he proposes to elucidate, together with a systematic classification of their types into series according to the number of their terms. He gives six binomial equations or simple equations as he calls them, and nineteen compound equations. The latter he subdivides into trinomial and tetranomial equations. The trinomial equations are the quadratic equations, cubic equations reducible to quadratic forms and proper cubic equations. The tetranomial equations are in turn subdivided into two groups, according to whether the sum of two terms is equal to the sum of the other two or the sum of two terms equal to the fourth.

His next two chapters present the algebraic solution of his simple and quadratic equations, paralleled by geometrical construction and demonstration of the same. In some cases he considers conditions of solubility and in some of them he gives more than one solution, with critical considerations of methods presented by earlier writers. The quadratic equations are followed by a presentation of such preliminary theorems

as he considered necessary for the construction of cubic equations.

The three succeeding chapters are devoted to the geometrical construction and demonstration of cubic equations of all types, trinomial and tetranomial. Here he abandons the rule of verifying his algebraic solutions by geometrical demonstrations which he sets for himself at the beginning of his work. But the interesting feature of this section is that he introduces his demonstrations by such lemmas and such properties of conic sections or reference to Apollonius's work on *Conics* and Euclid's *Elements* as he would need later in demonstrating his theorems.

The ninth chapter is a discussion of equations containing fractional terms. He uses for their denominators powers of the unknown. He reduces such equations to twenty-five primitive types, either by replacing the unknown by its reciprocal or by multiplying the proposed equation by a power of the unknown.

The tenth chapter which he wrote at the request of one of his friends, five years after the main work was completed, is an analysis and critical discussion of the work of Abu'l Jud on the demonstration of some of the cubic equations proposed by Omar himself in his present work.

In Kasir's translation, particular pains have been taken to adhere as closely as possible to the original. But in places where a too literal translation might confuse the English reader a free rendering of Arabic expressions is given so as to conform with modern phraseology. In such instances the text is often supplemented by an explanatory footnote. Also, to make it easier for the reader to follow, the algebraic solutions and the geometrical demonstrations given in the text in the usual rhetorical form of that period, are translated into modern symbolism. Further, the reader will find, in addition to this symbolic translation, a brief interpretation of Omar's method together with an account of the historic development of some of the particular discussions therein, tracing such conclusions to earlier writers whose work influenced the Persian Author. Finally, special effort has been made to point out in this treatise the distinctive contribution of Omar to the science of mathematics.

In addition to Professor Smith's manuscript the translator has used all available literature on the subject by European writers and historians of mathematics, particularly a translation of Omar Khayyam's algebra by the mathematician Woepcke. This was used for verifying a number of doubtful points in Professor Smith's manuscript. Also a generous use was made of all Arabic sources in which references to Omar's life and work could be found.

The general conclusion drawn from the study of Omar Khayyam's work is that if we are to consider his algebra from the standpoint of modern algebraic theory we find that it involves some defective conclusions and perhaps some avoidable shortcomings as, for example, in his neglect of negative roots, and multiple, fractional, and imaginary roots. He limited his results to a narrow domain of integers in case of algebraic solutions and to one or at most two roots in his geometrical constructions. The reason for this limitation is found in the incomplete geometrical representation of his constructions.

Turning to his principal achievements, we find that he made a remarkable advance toward the development of an algebraic science both by giving a complete classification of equations through the third degree on the basis

of the number of terms and by providing in his analytical solutions and geometrical constructions and demonstrations a systematic method of reasoning. The orderliness of his mind and the systematic character of his work are exemplified in his solutions of equations through the quadratic forms followed as they are by geometrical constructions and demonstrations for purposes of verification. In fact it is an effort to unify algebra and geometry. But his principal contribution is his classification and construction of cubic equations. He demonstrated each of his proposed equations of the third degree type and discussed the necessary modifications in the case of particular types.

It is hoped that this work will be of special value to those who are interested in the historic development of algebra as it gives a general account of the development of the science from the time of Diophantus to the time of Omar Khayyam. It points out the steps taken to transform the geometric algebra of the Greeks to the analytic algebra of the Arabs. The book should be of especial interest to all the Omar Khayyam clubs throughout the country.

*First Year Algebra.* By David A. Rothrock and Martha Annie Whitacre. Pages xi+320. Charles Scribner's Sons, 1932. Price $1.20.

This book is intended to present the essentials of algebra necessary to enable the pupil to appreciate and interpret the formulas of geometry, physics, and other sciences. It contains an abundance of drill material for giving him skill in the mechanics of algebra and many applied problems are included so as to enable the pupil to obtain an insight into the various applications of mathematics to everyday life.

Subjects not essential to the study and understanding of algebra have to some extent been omitted and new subject matter along professional lines has been included.

Problems have been grouped by types in order to stimulate the pupil's interest and to develop his confidence in analyzing the conditions of such problems so as to obtain the correct equations.

The review exercises at the end of each chapter serve as test material and furnish an opportunity to the pupil for maintaining skills developed earlier in the course.

*Agricultural Mathematics.* By L. C. Plant. Pages vi+199. McGraw-Hill Book Company, 1930. Price $2.50.

This book is a text on fundamental mathematics, emphasizing agricultural applications and the importance of mathematics as a tool in solving agricultural problems. It is the outgrowth of classroom experience and should be of interest to other teachers wishing workable material.

*Statistical Tables and Graphs.* By Bruce D. Mudgett. Pages viii+194. Houghton Mifflin Company, 1930. $1.75.

This book has grown out of ten years' experience in teaching elementary statistical methods to students preparing to enter the school of Business Administration at the University of Minnesota. While not all-inclusive it contains the most probably useful methods with which business men are likely to meet in daily life and since the author sticks to fundamental needs he is able to present the material in such a way as to make it accessible.